

Slope Fields

We will consider differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

Example

$$\frac{dy}{dt} = y - t.$$

A solution of this differential equation is

$$y = t + 1 - e^t.$$

Check: Let y be the function $y = t + 1 - e^t$.
Then

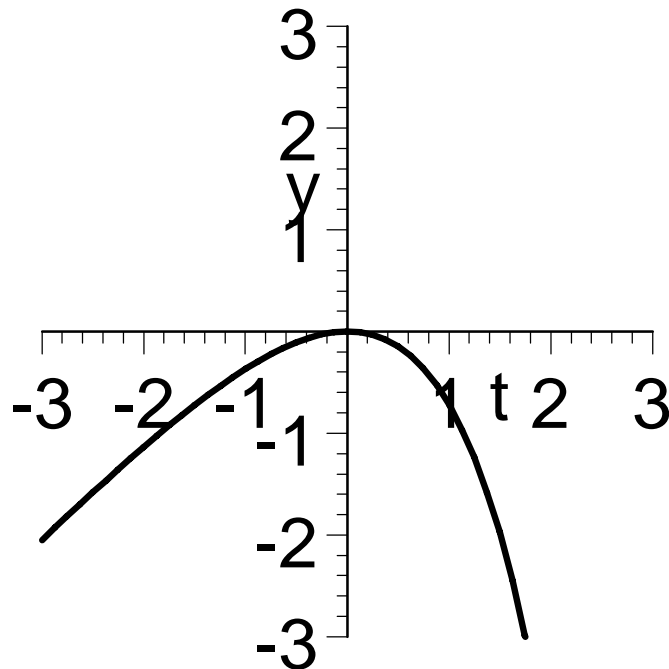
$$\frac{dy}{dt} = 1 - e^t$$

and

$$y - t = (t + 1 - e^t) - t = 1 - e^t$$

so $dy/dt = y - t$, which shows that the function $y = 1 + t - e^t$ is a solution of $dy/dt = y - t$.

Here is a graph of the solution $y = 1 + t - e^t$:



Recall that the value of dy/dt at any given point (t_1, y_1) on the graph of y is the slope of the tangent line to the graph of y at that point. For example,

- At $t = -1$, we have

$$y = -1 + 1 - e^{-1} = -e^{-1}$$

so the point $(-1, -e^{-1})$ lies on the graph of y .

At this point, we have

$$\begin{aligned}
\frac{dy}{dt} &= y - t \\
&= -e^{-1} - (-1) \\
&= 1 - e^{-1} \\
&\approx 0.63212
\end{aligned}$$

so the slope of the tangent line to the graph of y at the point $(-1, -e^{-1})$ is approximately 0.63212.

- At $t = 0$, we have

$$y = 0 + 1 - e^0 = 0$$

so the point $(0, 0)$ lies on the graph of y . At this point, we have

$$\begin{aligned}
\frac{dy}{dt} &= y - t \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

so the slope of the tangent line to the graph of y at the point $(0, 0)$ is 0.

- At $t = 1$, we have

$$y = 1 + 1 - e^1 = 2 - e$$

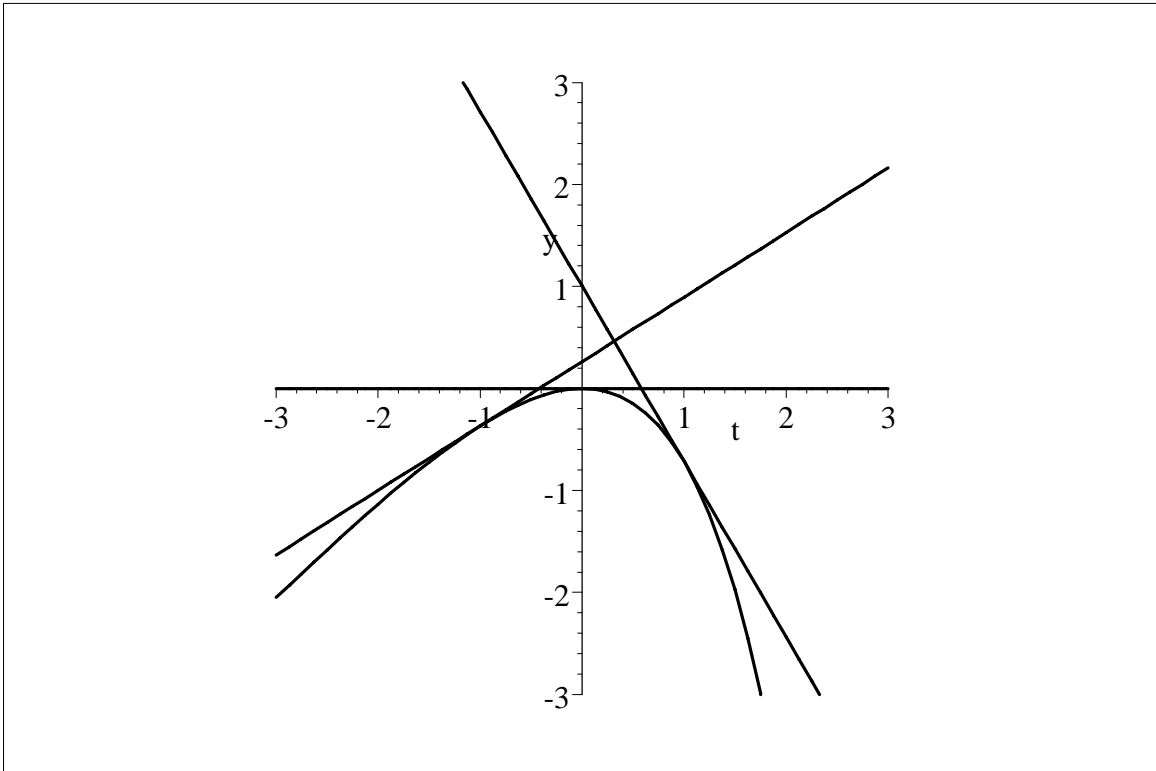
so the point $(1, 2 - e)$ lies on the graph of y .

At this point, we have

$$\begin{aligned}\frac{dy}{dt} &= y - t \\ &= (2 - e) - (1) \\ &= 1 - e \\ &\approx -1.7183\end{aligned}$$

so the slope of the tangent line to the graph of y at the point $(1, 2 - e)$ is approximately -1.7183 .

The following picture shows the graph of y along with graphs of the three tangent lines whose slopes were just computed.



The main thing to observe here is that if we pick *any* point (t_1, y_1) in the plane, we can easily compute the value of $y_1 - t_1$. Thus, *without knowing* the solution of the differential equation

$$\frac{dy}{dt} = y - t$$

whose graph passes through the point (t_1, y_1) , we *do know* that the slope of the tangent line to this graph at the point (t_1, y_1) is $y_1 - t_1$.

For example, suppose that we pick the point $(0, 3)$ in the (t, y) plane. At this point, we have

$$y - t = 3 - 0 = 3$$

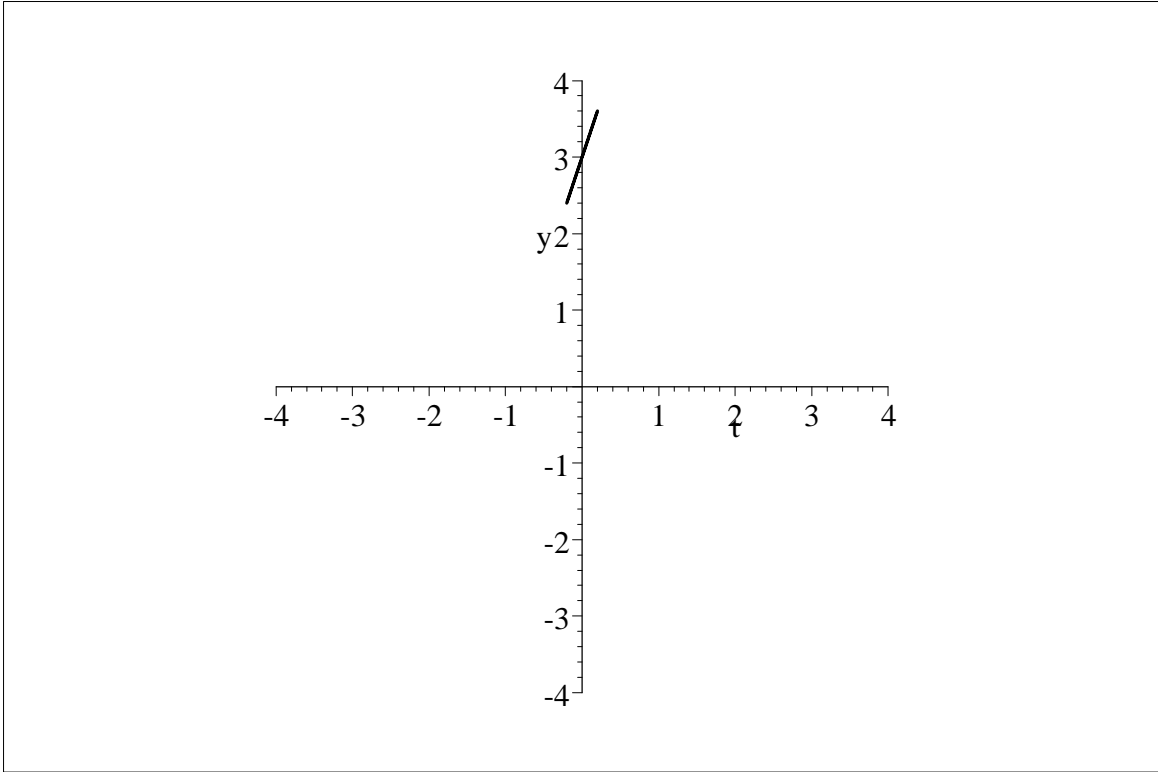
which tells us that 3 is the slope of the tangent line to the graph of the solution of the differential equation

$$\frac{dy}{dt} = y - t$$

whose graph passes through the point $(0, 3)$. We can draw a “minitangent” line with slope 3 at the point $(0, 3)$. This minitangent line gives us visual information about the graph of the solution of the initial value problem

$$\frac{dy}{dt} = y - t$$

$$y(0) = 3$$



The actual solution of the initial value problem

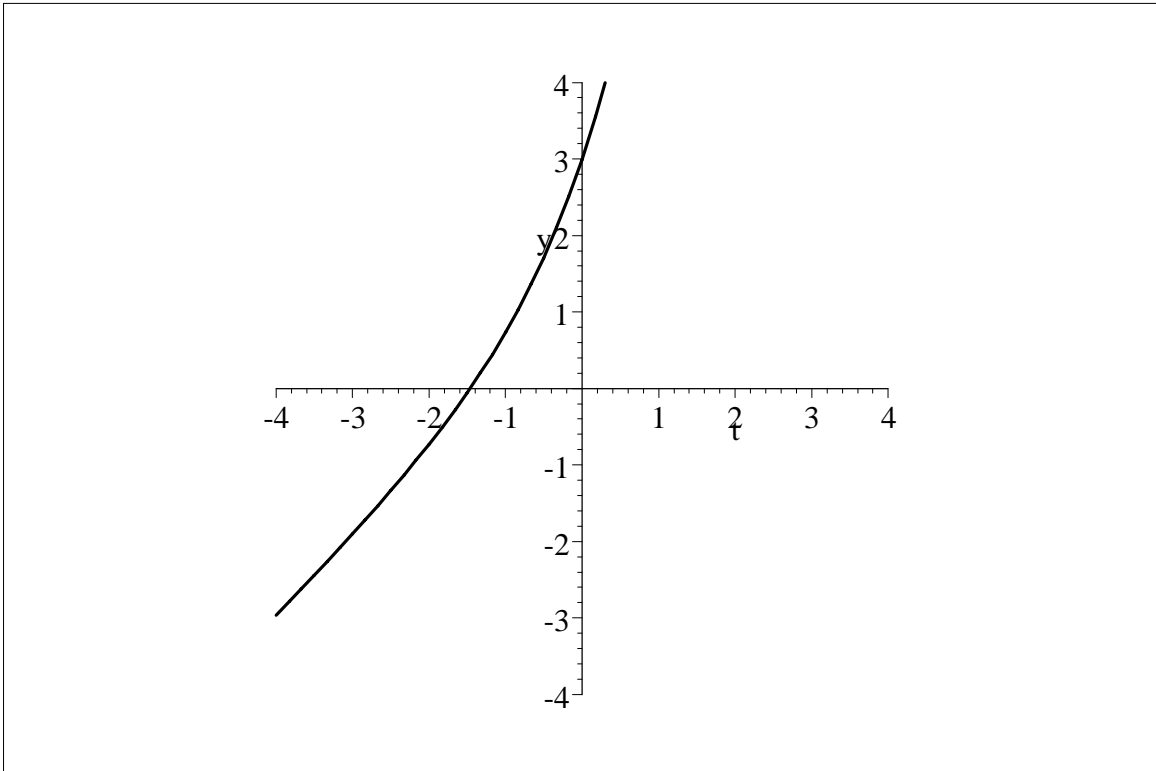
$$\frac{dy}{dt} = y - t$$

$$y(0) = 3$$

is

$$y = t + 1 + 2e^t.$$

The graph of this solution is shown below.



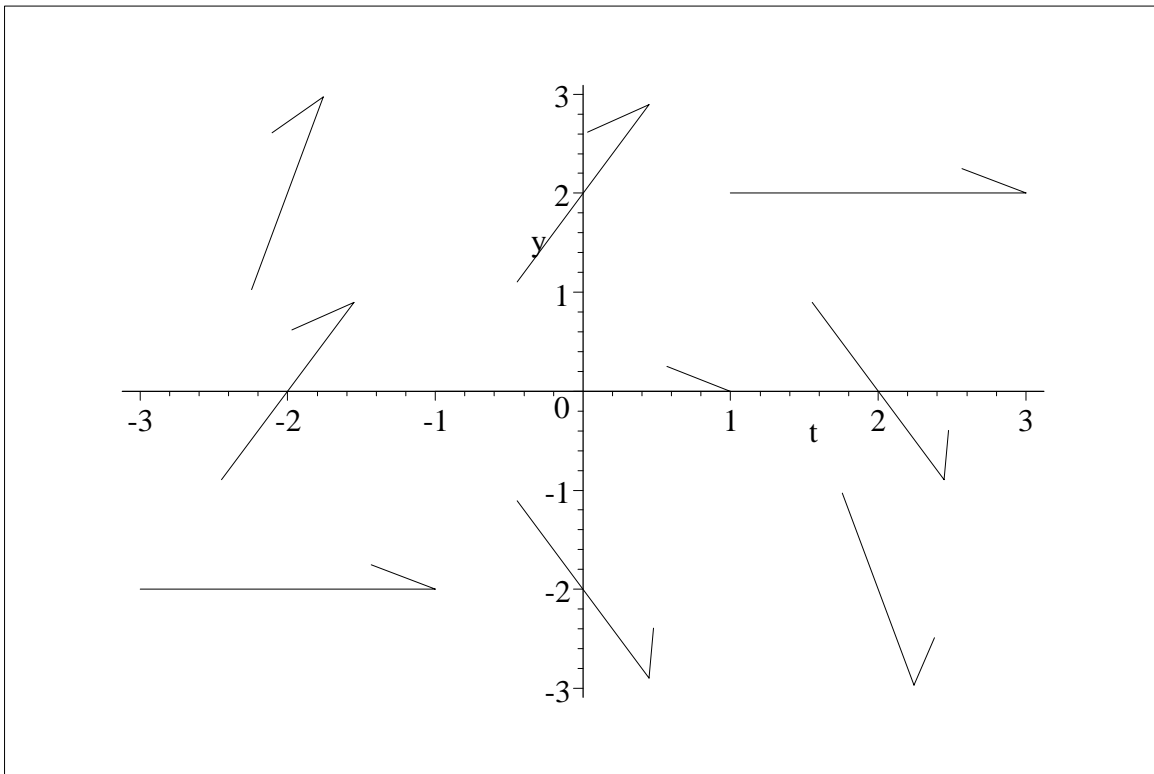
If we pick a large number of points in the (t, y) plane, compute the value of $y - t$, and draw minitangents at these points, we get a visual picture of what the entire *family* of solutions of

$$\frac{dy}{dt} = y - t$$

looks like. What is nice about this is that it is very *easy*. We can get a good picture of what solutions look like without actually knowing (explicitly) what the solutions are. Below is a table with slopes of minitangents computed at selected points and a picture showing these minitangents. This type of picture is called a

slope field.

(t, y)	$y - t$	(t, y)	$y - t$	(t, y)	$y - t$
$(-2, 2)$	4	$(0, 2)$	2	$(2, 2)$	0
$(-2, 0)$	2	$(0, 0)$	0	$(2, 0)$	-2
$(-2, -2)$	0	$(0, -2)$	-2	$(2, -2)$	-4

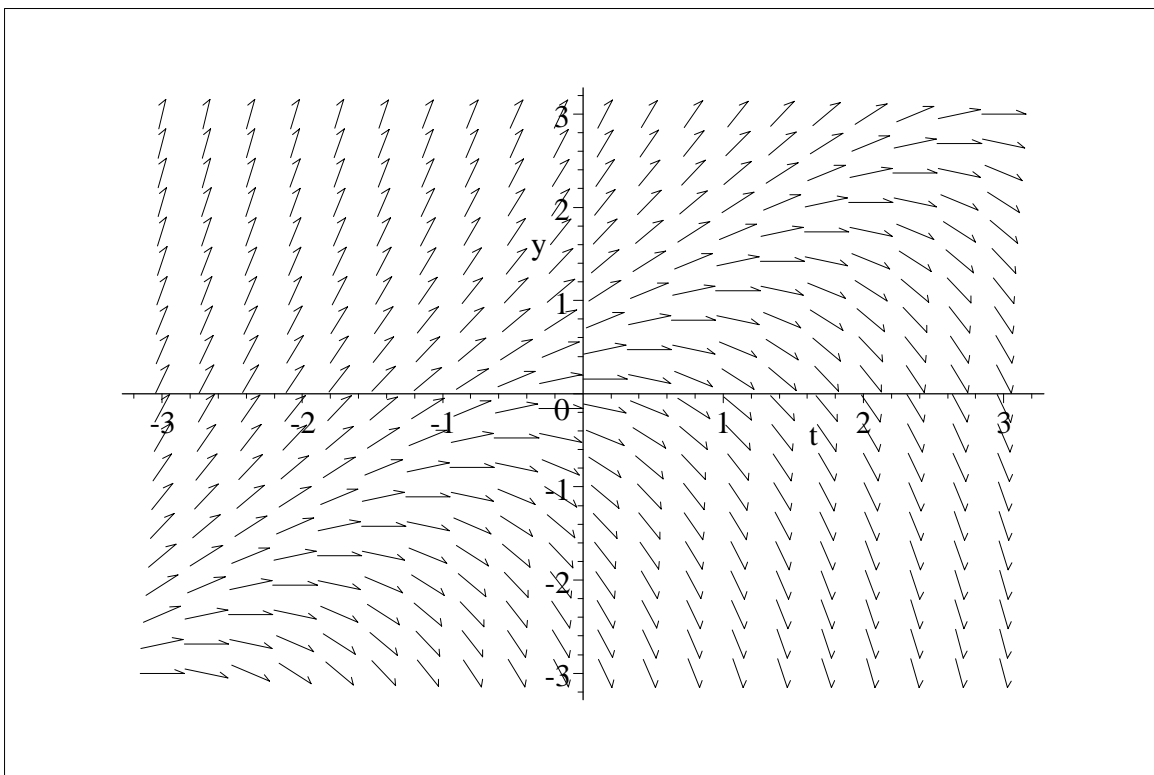


Of course, to get a really good picture of the behavior of solutions, we need to compute $y - t$ at many points. This is easily handled by a computer with a program written to compute and draw slope fields. Here is a

computer-generated picture of the slope field of

$$\frac{dy}{dt} = y - t$$

using a much larger sampling of points in the (t, y) plane.



By looking at the above computer-generated slope field, we can visualize the family of solutions of $dy/dt = y - t$. In particular, note that there appears to be a solution whose graph is a line with slope 1 passing through the point $(0, 1)$. This line is, of course, $y = t + 1$. It is easy to check that the function

$y = t + 1$ is indeed a solution of the differential equation $dy/dt = y - t$.

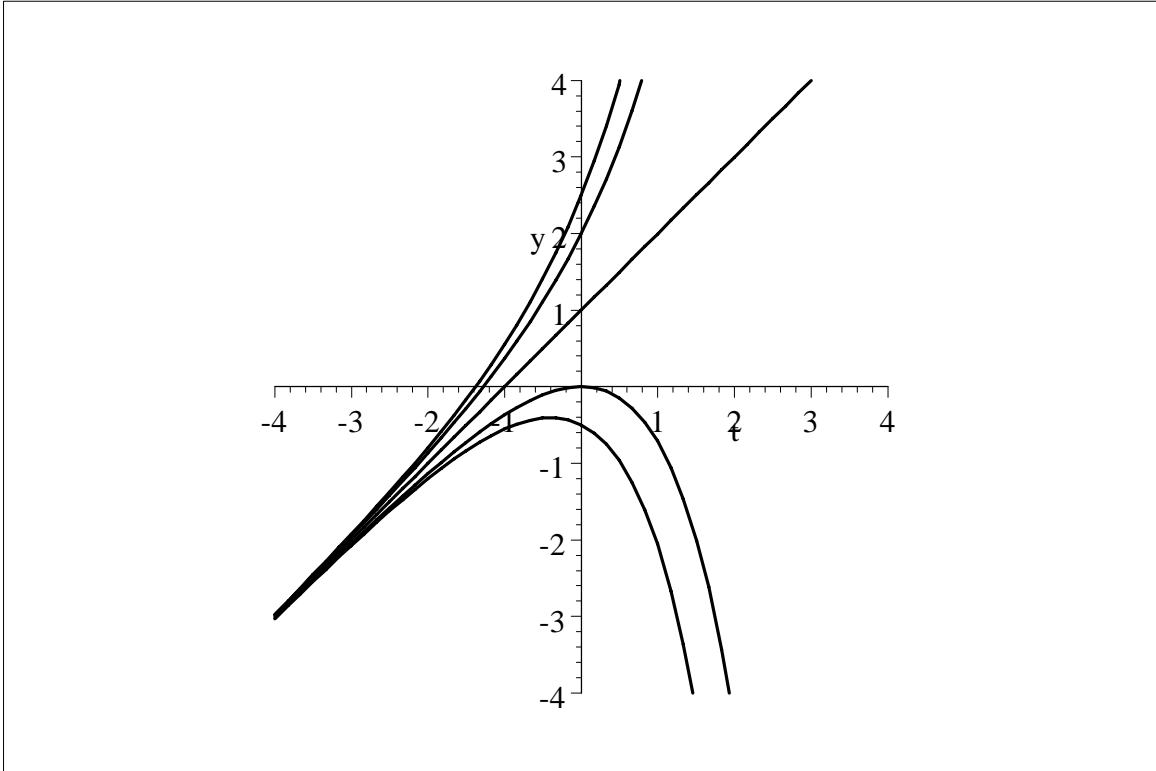
Furthermore, it appears from the slope field that solutions whose graphs lie above the line $y = t + 1$ “blow up” as $t \rightarrow \infty$ (meaning that $\lim_{t \rightarrow \infty} y = \infty$) and approach the line $y = t + 1$ as $t \rightarrow -\infty$ (meaning that $\lim_{t \rightarrow -\infty} (y - (t + 1)) = 0$) and it appears that solutions whose graphs lie below the line $y = t + 1$ “blow up” as $t \rightarrow \infty$ (in this case, in the sense that $\lim_{t \rightarrow \infty} y = -\infty$) and approach the line $y = t + 1$ as $t \rightarrow -\infty$.

In fact, the family of solutions of $dy/dt = y - t$ consists of all functions of the form

$$y = t + 1 + ce^t$$

(where c can be any constant). The value $c = 0$ gives us the solution $y = t + 1$. Solutions whose graphs lie above the line $y = t + 1$ correspond to positive values of c and solutions whose graphs lie below the line $y = t + 1$ correspond to negative values of c . The figure below shows graphs of some members of this family (corresponding to the

choices $c = 0$, $c = 1$, $c = 1.5$, $c = -1$, and $c = -1.5$).



Summary of the General Idea About Slope Fields

Given a differential equation of the form

$$\frac{dy}{dt} = f(t, y),$$

we can choose any point, (t_1, y_1) , in the (t, y) plane and compute the value of $f(t_1, y_1)$. The value of $f(t_1, y_1)$ is the slope of the line tangent to the graph of the function, y , that satisfies

$$\frac{dy}{dt} = f(t, y)$$

$$y(t_1) = y_1.$$

If we compute the value of $f(t, y)$ at many different points in the (t, y) plane and draw minitangents with these slopes at these points, then we will have obtained a *slope field* for the differential equation

$$\frac{dy}{dt} = f(t, y).$$

The slope field gives us a visual picture of what the graphs of solutions look like.

A special case: Differential equations whose right hand side does not depend on y

If the right hand side of

$$\frac{dy}{dt} = f(t, y)$$

does not depend on y , then the minitangents which make up the slope field of this differential equation have the same slope

along any given vertical line. Hence, these slope fields are somewhat easier to create and visualize.

Example

Let us create a slope field for the differential equation

$$\frac{dy}{dt} = \frac{1}{2}t^2.$$

The right hand side of this differential equation is $f(t, y) = \frac{1}{2}t^2$ (which does not depend on y). Observe that

$$f(-2, 0) = \frac{1}{2}(-2)^2 = 2$$

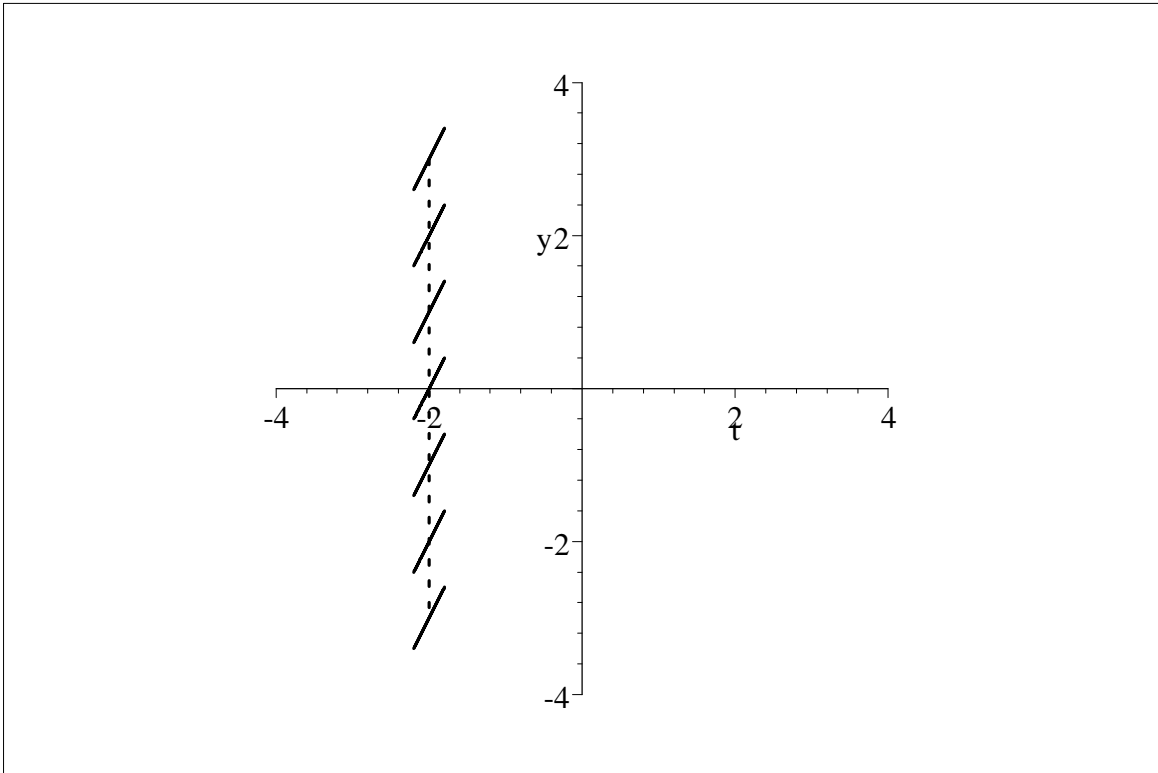
$$f(-2, 12) = \frac{1}{2}(-2)^2 = 2$$

$$f(-2, -\sqrt{17}) = \frac{1}{2}(-2)^2 = 2$$

etc.

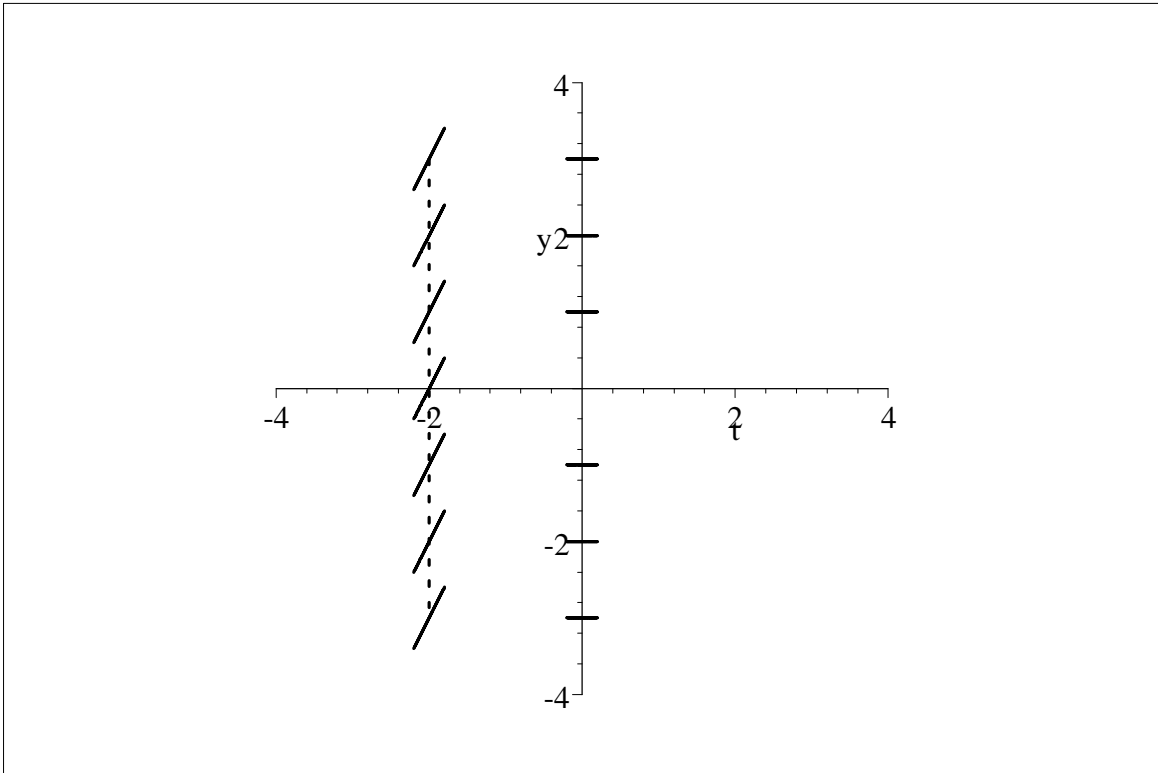
meaning that the slope of any minitangent that lies along the vertical line $t = -2$ has slope 2. Hence, we only really need to do one computation using $t = -2$:

$$f(-2, y) = 2 \quad \text{for all } y.$$



A similar result holds for any chosen value of t . For example,

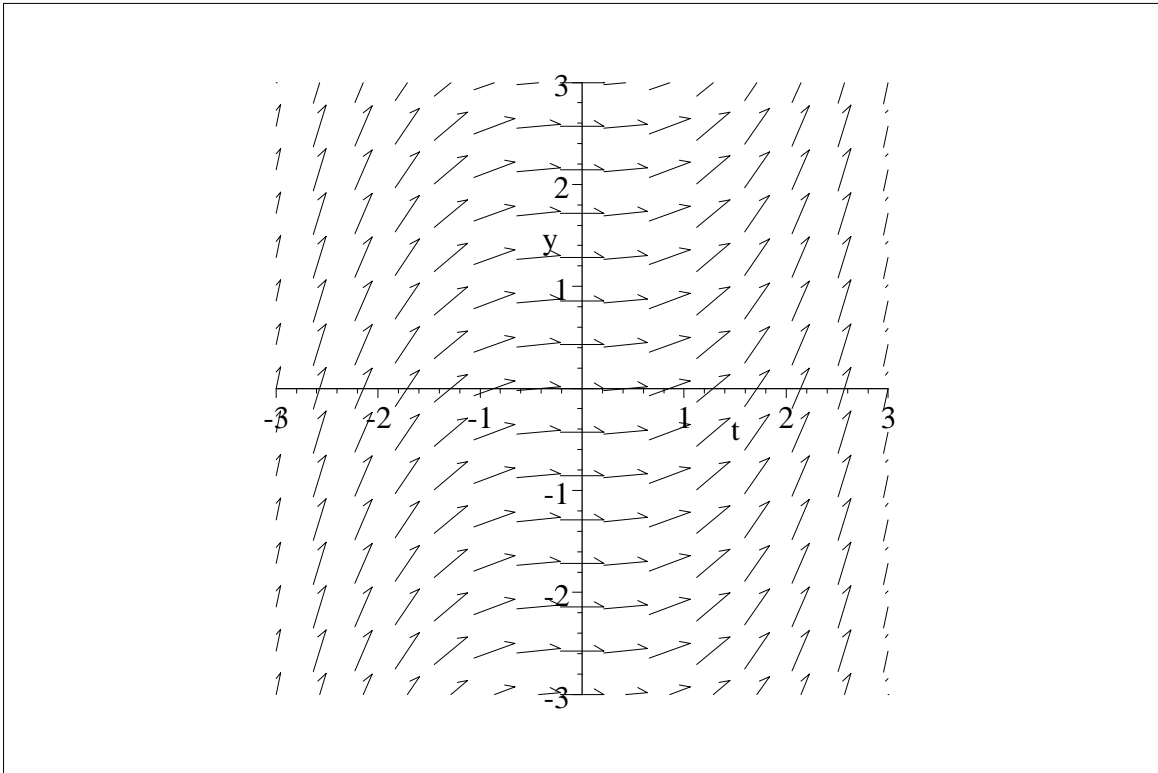
$$f(0, y) = 0 \quad \text{for all } y.$$



A computer-generated graph of the slope field of

$$\frac{dy}{dt} = \frac{1}{2}t^2$$

is shown below.



Of course, we can easily deduce (by antidifferentiation) that the family of solutions of the differential equation

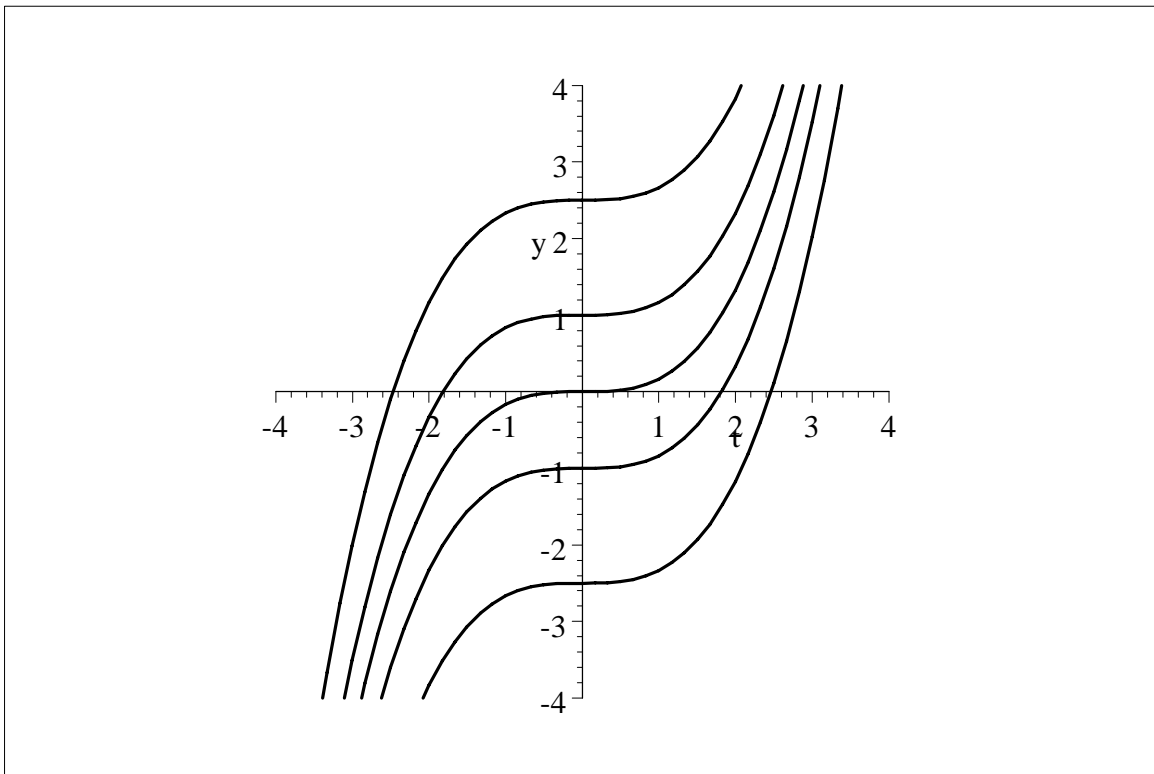
$$\frac{dy}{dt} = \frac{1}{2}t^2$$

consists of all functions of the form

$$y = \frac{1}{6}t^3 + C$$

(where C can be any constant). The fact that the graphs of solutions are all vertical translations of each other is evident from the slope field. Some members of the family of solutions are pictured below. (What values of

C were used to obtain the family members in the picture?)



A special case: Differential equations whose right hand side does not depend on t

If the right hand side of

$$\frac{dy}{dt} = f(t, y)$$

does not depend on t , then the minitangents which make up the slope field of this

differential equation have the same slope along any given horizontal line. Differential equations whose right hand sides do not depend on t are called *autonomous* differential equations. The word autonomous means “not depending on time”. Autonomous differential equations are used to model systems for which the rules (hypotheses) governing the system do not depend on what time it is. For example, the Malthusian population model,

$$\frac{dP}{dt} = kP$$

is autonomous because the rule governing the system is simply that a population always grows at a rate proportional to itself. The model predicts, for example, that the growth history of a population that starts with 3 million individuals in the year 1790 would be the same as the growth history of a population that starts with 3 million individuals in the year 1985. Both would, for instance, have the same number of individuals after 20 years.

Example

Let us create a slope field for the autonomous differential equation

$$\frac{dy}{dt} = 0.2y\left(1 - \frac{y}{6}\right).$$

The right hand side of this differential equation is $f(t, y) = 0.2y(1 - y/6)$ (which does not depend on t). After doing a few computations:

$$f(t, -2) = -\frac{8}{15}$$

$$f(t, -1) = -\frac{7}{30}$$

$$f(t, 0) = 0$$

$$f(t, 1) = \frac{1}{6}$$

$$f(t, 2) = \frac{4}{15}$$

$$f(t, 3) = \frac{3}{10}$$

$$f(t, 4) = \frac{4}{15}$$

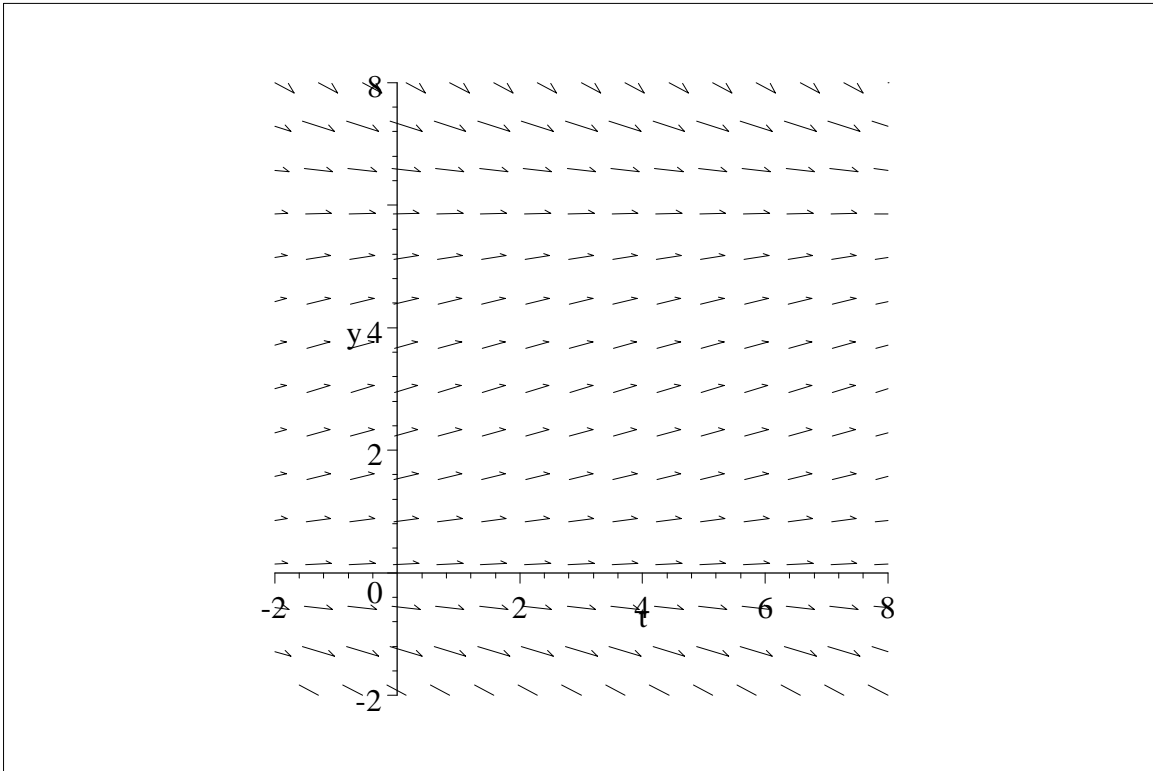
$$f(t, 5) = \frac{1}{6}$$

$$f(t, 6) = 0$$

$$f(t, 7) = -\frac{7}{30}$$

$$f(t, 8) = -\frac{8}{15}$$

we observe that the slope field of this differential equation has the appearance shown in the following figure.



Note that the slopes of minitangents are the same along any given horizontal line. In particular, note that since $f(t, 0) = 0$ and $f(t, 6) = 0$ for any t , there are constant solutions that lie along the lines $y = 0$ and $y = 6$. These solutions are called *equilibrium solutions* and are a common feature of autonomous differential equations.