

1 Introduction

Named after Joseph Fourier (1768-1830).

Like Taylor series, they are special types of expansion of functions.

Taylor series: we expand a function in terms of the special set of functions $1, x, x^2, x^3, \dots$ or more generally in terms of $1, (x - a), (x - a)^2, (x - a)^3$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \quad (1)$$

Fourier series: we expand a function f in terms of the special set of functions $1, \cos x, \cos 2x, \cos 3x, \dots, \sin x, \sin 2x, \sin 3x, \dots$. Thus, a Fourier series expansion of a function is an expression of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

2 Even, Odd and Periodic Functions

Definition 1 (Even and Odd) *Let f be a function defined on an interval I (finite or infinite) centered at $x = 0$.*

1. f is said to be **even** if $f(-x) = f(x)$ for every x in I .
2. f is said to be **odd** if $f(-x) = -f(x)$ for every x in I .

The graph of an even function is symmetric with respect to the y -axis. The graph of an odd function is symmetric with respect to the origin. For example, 5 , x^2 , x^n where n is even, $\cos x$ are even functions while x , x^3 , x^n where n is odd, $\sin x$ are odd.

Theorem 2 *Let f be a function which domain includes $[-a, a]$ where $a > 0$.*

1. *If f is even, then $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$*
2. *If f is odd, then $\int_{-a}^a f(x) dx = 0$*

There are several useful algebraic properties of even and odd functions as shown in the theorem below.

Theorem 3 *When adding or multiplying even and odd functions, the following is true:*

- *even + even = even*
- *even \times even = even*
- *odd + odd = odd*

- $odd \times odd = even$
- $even \times odd = odd$

Definition 4 (Periodic) *Let $T > 0$.*

1. *A function f is called **T -periodic** or simply **periodic** if*

$$f(x + T) = f(x) \quad (2)$$

for all x .

2. *The number T is called a **period** of f .*

3. *If f is non-constant, then the smallest positive number T with the above property is called the **fundamental period** or simply the **period** of f .*

Let us first remark that if T is a period for f , then nT is also a period for any integer $n > 0$.

Classical examples of periodic functions are $\sin x$, $\cos x$ and other trigonometric functions. $\sin x$ and $\cos x$ have period 2π . $\tan x$ has period π .

Because the values of a periodic function of period T repeat every T units, it is enough to know such a function on any interval of length T . Its graph is obtained by repeating the portion over any interval of length T . Consequently, to define a T -periodic function, it is enough to define it over any interval of length T . Since different intervals may be chosen, the same function may be defined different ways.

Example 5 Describe the 2-periodic function shown in figure 1 in two different ways:

1. By considering its values on the interval $0 \leq x < 2$;
2. By considering its values on the interval $-1 \leq x < 1$.

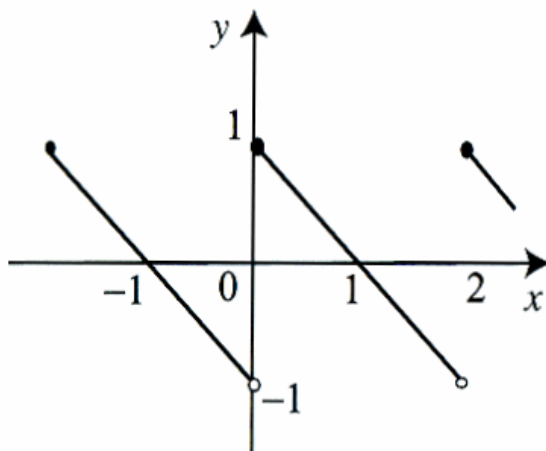


Figure 1: A function of period 2

Next, we look at an important theorem concerning integration of periodic functions over one period.

Theorem 6 (Integration Over One Period) *Suppose that f is T -periodic. Then for any real number a , we have*

$$\int_0^T f(x) dx = \int_a^{a+T} f(x) dx \quad (3)$$

We illustrate this theorem with an example.

Example 7 *Let f be the 2-periodic function shown in figure 1. Compute the integrals below:*

1. $\int_{-1}^1 [f(x)]^2 dx$
2. $\int_{-N}^N [f(x)]^2 dx$ where N is any positive integer.

The following result about combining periodic functions is important.

Theorem 8 *When combining periodic functions, the following is true:*

1. *If f_1, f_2, \dots, f_n are T -periodic, then $a_1f_1 + a_2f_2 + \dots + a_nf_n$ is also T -periodic.*
2. *If f and g are two T -periodic functions so is $f(x)g(x)$.*
3. *If f and g are two T -periodic functions so is $\frac{f(x)}{g(x)}$ where $g(x) \neq 0$.*
4. *If f has period T and $a > 0$ then $f\left(\frac{x}{a}\right)$ has period aT and $f(ax)$ has period $\frac{T}{a}$.*
5. *If f has period T and g is any function (not necessarily periodic) then the composition $g \circ f$ has period T .*

The functions in the 2π -periodic trigonometric system

$1, \cos x, \cos 2x, \dots, \cos mx, \dots, \sin x, \sin 2x, \dots, \sin nx, \dots$

are among the most important periodic functions. The reader will verify that they are indeed 2π -periodic. They share another important property.

Definition 9 (Orthogonal Functions) *Two functions f and g are said to be **orthogonal** over the interval $[a, b]$ if*

$$\int_a^b f(x) g(x) dx = 0 \quad (4)$$

The notion of orthogonality is very important in many areas of mathematics.

Theorem 10 *The functions 1, $\cos mx$, $\sin nx$ are orthogonal over the interval $[-\pi, \pi]$ that is if m and n are two nonnegative integers, then*

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx dx &= 0 \text{ if } m \neq n & (5) \\ \int_{-\pi}^{\pi} \cos mx \sin nx dx &= 0 \quad \forall m, n \\ \int_{-\pi}^{\pi} \sin mx \sin nx dx &= 0 \text{ if } m \neq n \end{aligned}$$

Important identities:.

$$\begin{aligned} \sin \alpha \cos \beta &= \frac{1}{2} [\sin (\alpha + \beta) + \sin (\alpha - \beta)] \\ \cos \alpha \sin \beta &= \frac{1}{2} [\sin (\alpha + \beta) - \sin (\alpha - \beta)] \\ \sin \alpha \sin \beta &= \frac{1}{2} [\cos (\alpha + \beta) - \cos (\alpha - \beta)] \\ \cos \alpha \cos \beta &= \frac{1}{2} [\cos (\alpha + \beta) + \cos (\alpha - \beta)] \end{aligned}$$

$$\int_{-\pi}^{\pi} \cos^2 mx dx = \int_{-\pi}^{\pi} \sin^2 mx dx = \pi \text{ for all } m \neq 0 \quad (6)$$

We finish this section by looking at another example of a periodic function, which does not involve trigonometric functions but rather the **greatest integer** function, also known as the **floor** function, denoted $\lfloor x \rfloor$. $\lfloor x \rfloor$ represents the greatest integer not larger than x . For example, $\lfloor 5.2 \rfloor = 5$, $\lfloor 5 \rfloor = 5$, $\lfloor -5.2 \rfloor = -6$, $\lfloor -5 \rfloor = -5$. Its graph is shown in figure 2.

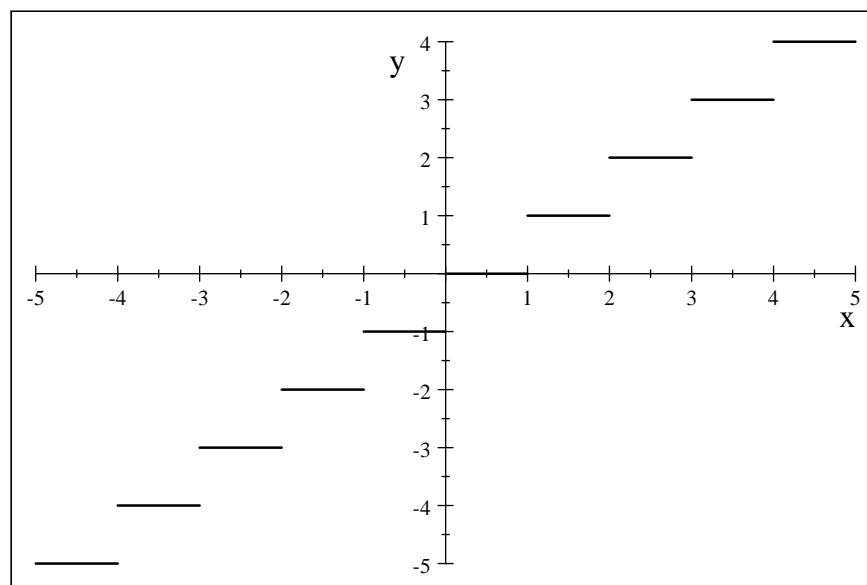


Figure 2: Graph of $\lfloor x \rfloor$

Example 11 Let $f(x) = x - \lfloor x \rfloor$. This gives the fractional part of x . For $0 \leq x < 1$, $\lfloor x \rfloor = 0$, so $f(x) = x$.

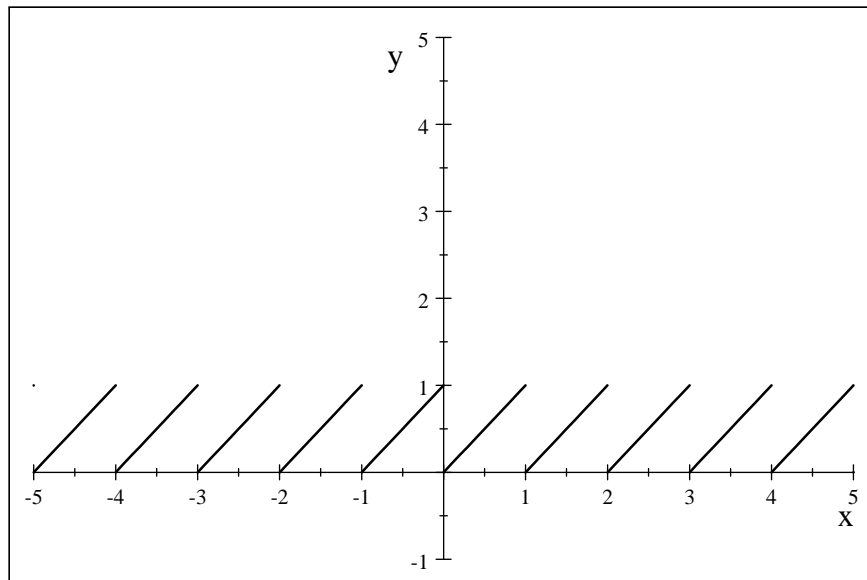


Figure 3: Graph of $x - [x]$

Also, since $[x + 1] = 1 + [x]$, we get

$$\begin{aligned}
 f(x + 1) &= x + 1 - [x + 1] \\
 &= x + 1 - 1 - [x] \\
 &= x - [x] \\
 &= f(x)
 \end{aligned}$$

So, f is periodic with period 1. Its graph is obtained by repeating the portion of its graph over the interval $0 \leq x < 1$. Its graph is shown in figure

3 Fourier Series of 2π -Periodic Functions

As noted earlier, Fourier Series are special expansions of functions of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (7)$$

To be able to use Fourier Series, we need to know:

1. Which functions have Fourier series expansions?
2. If a function has a Fourier series expansion, how do we compute the coefficients $a_0, a_1, \dots, b_0, b_1, \dots$?

A general answer to the first question is beyond the scope of these notes. In this section, we will answer the second question. In these notes, we will give conditions which are sufficient for functions to have a Fourier Series Expansion.

3.1 Euler Formulas for the Coefficients

The coefficients which appear in the Fourier series were known to Euler before Fourier, hence they bear his name. We will derive them the same way Fourier did. This technique is worth remembering.

Proposition 12 *Suppose that the 2π -periodic function f has the Fourier series representation*

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Then the coefficients a_0, a_n, b_n for $n = 1, 2, \dots$ are called the Fourier coefficients of f and are given by the Euler's formulas

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \quad (8)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \text{ for } n = 1, 2, \dots \quad (9)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \text{ for } n = 1, 2, \dots \quad (10)$$

Definition 13 For a positive integer N , we denote the N^{th} partial sum of the Fourier series of f by $S_N(x)$. So, we have

$$S_N(x) = a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx)$$

Example 14 Find the Fourier series of $f(x) = \sin x$.

Example 15 Find the Fourier series of $f(x) = \left| \sin \frac{x}{2} \right|$. Clearly, this function is 2π -periodic. Its graph is shown in figure 4.

$$\left| \sin \frac{x}{2} \right| = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{-4}{\pi(4n^2 - 1)} \cos nx$$

To see how this series compares to the function, we will plot some of the partial sums. Let

$$S_N(x) = \frac{2}{\pi} + \sum_{n=1}^N \frac{-4}{\pi(4n^2 - 1)} \cos nx$$

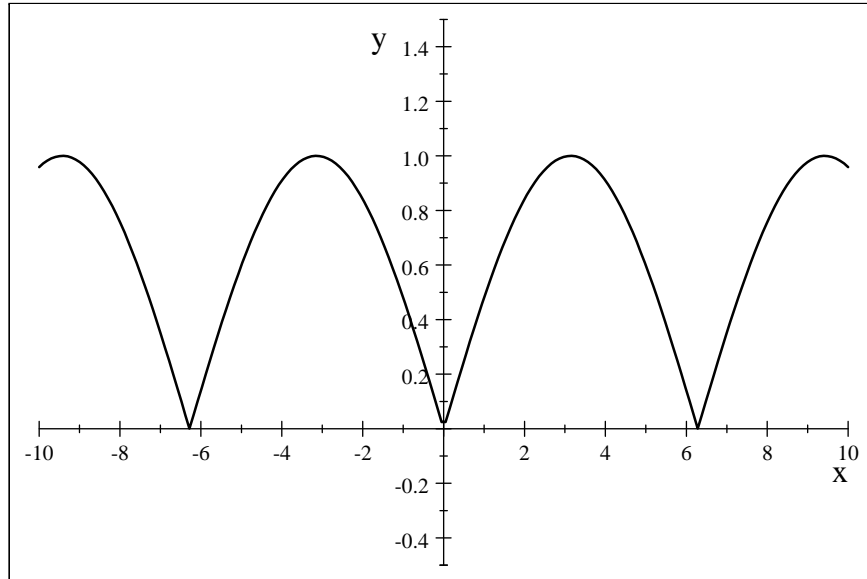


Figure 4: Graph of $|\sin x|$

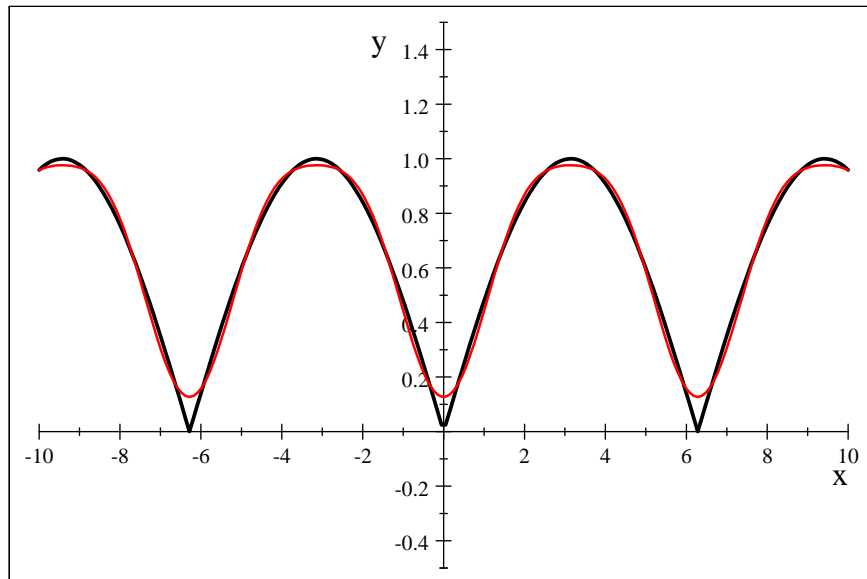


Figure 5: Graph of $|\sin \frac{x}{2}|$ and $S_2(x)$

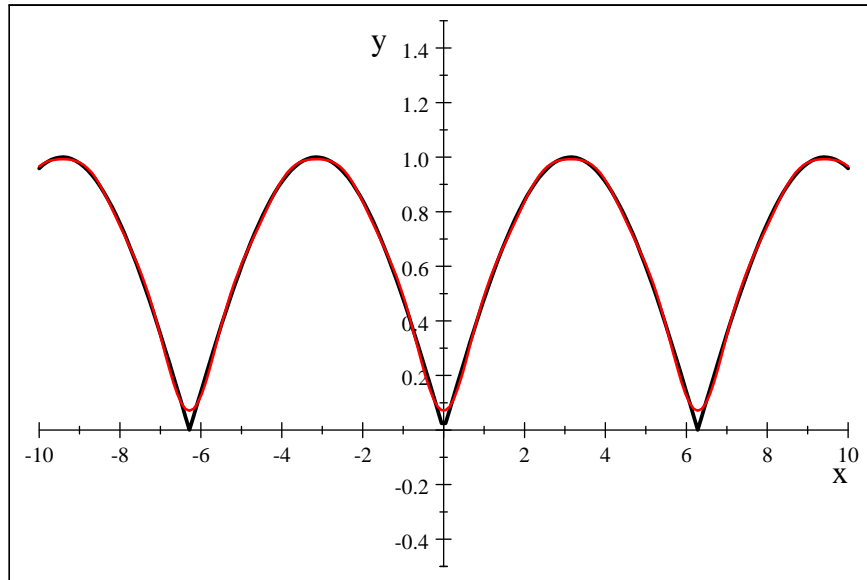


Figure 6: Graph of $|\sin \frac{x}{2}|$ and $S_4(x)$

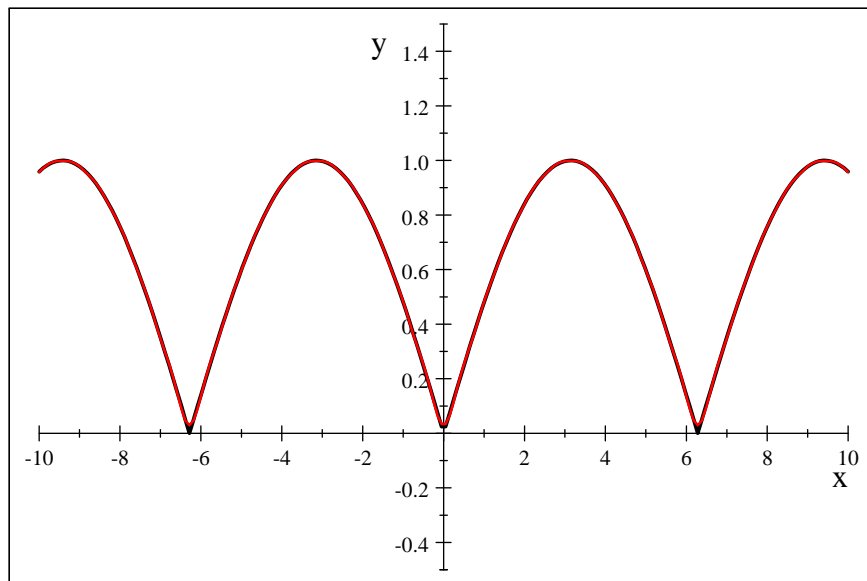


Figure 7: Graph of $|\sin \frac{x}{2}|$ and $S_{10}(x)$

Example 16 We now look at a 2π -periodic function with discontinuities and derive its Fourier series using the formulas of this section (assuming it is legitimate). This function is called the sawtooth function. It is defined by

$$g(x) = \begin{cases} \frac{1}{2}(\pi - x) & \text{if } 0 < x \leq 2\pi \\ g(x + 2\pi) & \text{otherwise} \end{cases}$$

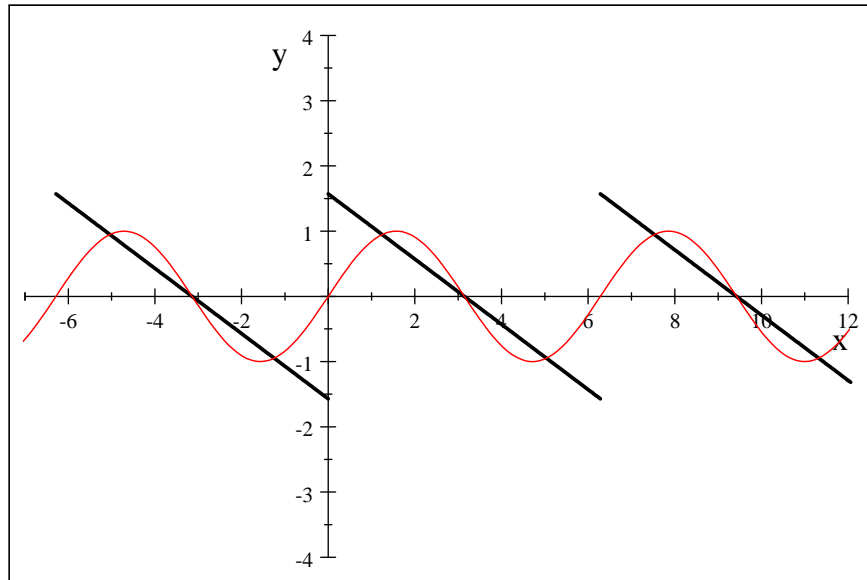
Find the Fourier series for this function. Plot this function as well as $S_1(x)$, $S_7(x)$, $S_{20}(x)$ where $S_N(x)$ is the N^{th} partial sum of its Fourier series.

Since f is described between 0 and 2π , we can use theorem 6 to compute the Fourier coefficients integrating between 0 and 2π .

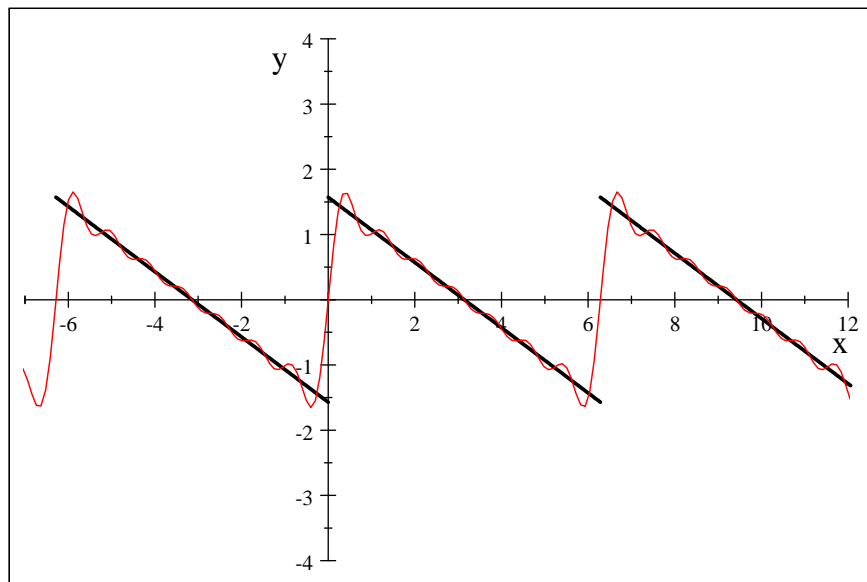
1. The Fourier series of the sawtooth function is

$$g(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

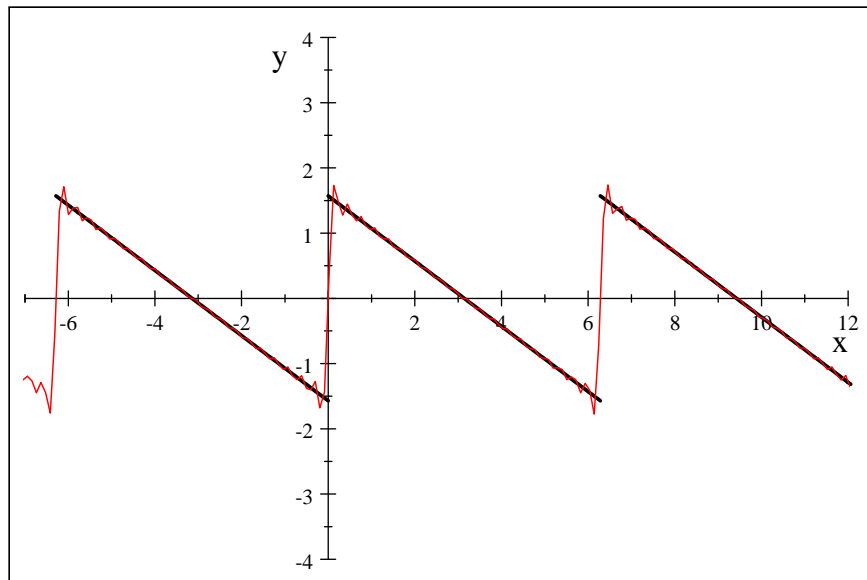
Below, we show the graphs of $S_1(x)$, $S_7(x)$, $S_{20}(x)$.



Graph of the sawtooth function (black) and $S_1(x)$ (red)



Graph of the sawtooth function (black) and $S_7(x)$ (red)



Graph of the sawtooth function (black) and $S_{20}(x)$ (red)

Remark 17 *Several important facts are worth noticing here.*

- 1. The Fourier series seems to agree with the function, except at the points of discontinuity.*
- 2. At the points of discontinuity, the series converges to 0, which is the average value of the function from the left and from the right.*

3. Near the points of discontinuity, the Fourier series overshoots its limiting values. This is a well known phenomenon, known as **Gibbs phenomenon**. To see a simulation of this phenomenon, visit the site <http://ocw.mit.edu/ans7870/18/18.06/javademo/Gibbs/>

3.2 Piecewise Continuous and Piecewise Smooth Functions

After defining some useful concepts, we give a sufficient condition for a function to have a Fourier series representation.

Notation 18 We will denote $f(c-) = \lim_{x \rightarrow c^-} f(x)$ and $f(c+) = \lim_{x \rightarrow c^+} f(x)$

Remembering that a function f is continuous at c if $\lim_{x \rightarrow c} f(x) = f(c)$, we see that a function f is continuous at c if and only if

$$f(c-) = f(c+) = f(c)$$

Definition 19 (Piecewise Continuous) A function f is said to be piecewise continuous on the interval $[a, b]$ if the following are satisfied:

1. $f(a+)$ and $f(b-)$ exist.
2. f is defined and continuous on (a, b) except at a finite number of points in (a, b) where the left and right limit at these points exist.

Definition 20 (Piecewise Smooth) A function f , defined on $[a, b]$ is said to be piecewise smooth if f and f' are piecewise continuous on $[a, b]$ that is if the following are satisfied:

1. f is piecewise continuous on $[a, b]$
2. f' exists on (a, b) except possibly at finitely many points in (a, b) where the one sided limits of f' at these points exist.
3. $\lim_{x \rightarrow a^+} f'(x)$ and $\lim_{x \rightarrow b^-} f'(x)$ exists.

The sawtooth function is piecewise smooth. A simple example of a function which is not piecewise smooth is $x^{\frac{1}{3}}$ for $-1 \leq x \leq 1$. Its derivative does not exist at 0, neither do the one sided limits of its derivative at 0.

Definition 21 *The average of f at c is defined to be*

$$\frac{f(c-) + f(c+)}{2}$$

Clearly, if f is continuous at c , then its average at c is $f(c)$.

We are now ready to state a fundamental result in the theory of Fourier series.

Theorem 22 *Suppose that f is a 2π -periodic piecewise smooth function. Then, for all x , we have*

$$\frac{f(x-) + f(x+)}{2} = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (11)$$

where the coefficients are given by equations 8, 9, and 10. In particular, if f is piecewise smooth and continuous at x , then

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (12)$$

Thus, at points where f is continuous, the Fourier series converges to the function. At points of discontinuity, the series converges to the average of the function at these points. This was the case in the example with the sawtooth function.

We do one more example.

Example 23 (Triangular Wave) *The 2π -periodic triangular wave is given on the interval $[-\pi, \pi]$ by*

$$h(x) = \begin{cases} \pi + x & \text{if } -\pi \leq x \leq 0 \\ \pi - x & \text{if } 0 \leq x \leq \pi \end{cases}$$

1. Find its Fourier series.

2. Plot $h(x)$ as well as some partial sums of its Fourier series.
3. Show how this series could be used to approximate π (actually π^2).

Solution 2A We begin by plotting $h(x)$ We see the func-

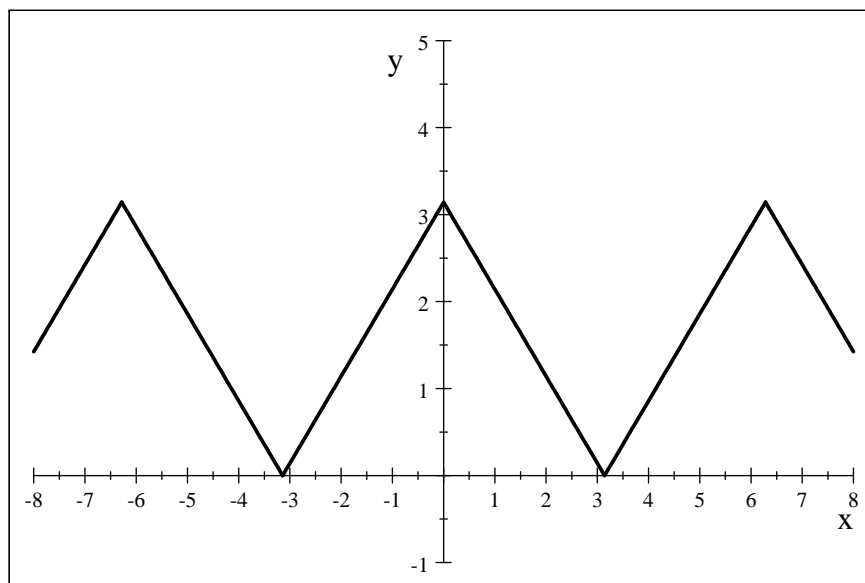


Figure 8: Plot of the triangular wave
tion is piecewise smooth and continuous for all x . Its

Fourier series is:

$$h(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$$

2. Let $S_N(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^N \frac{\cos(2n+1)x}{(2n+1)^2}$. We plot $S_1(x)$,
 $S_5(x)$

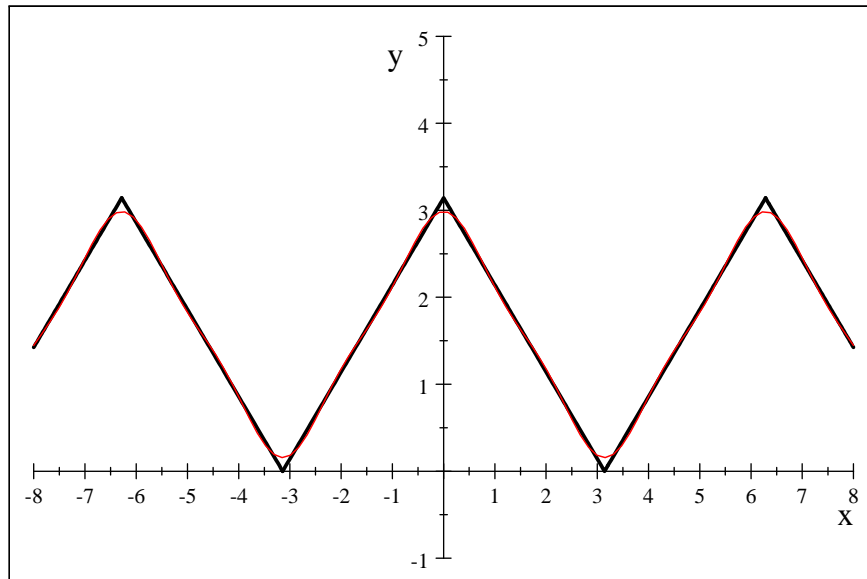


Figure 9: Plot of the triangular wave and $S_1(x)$

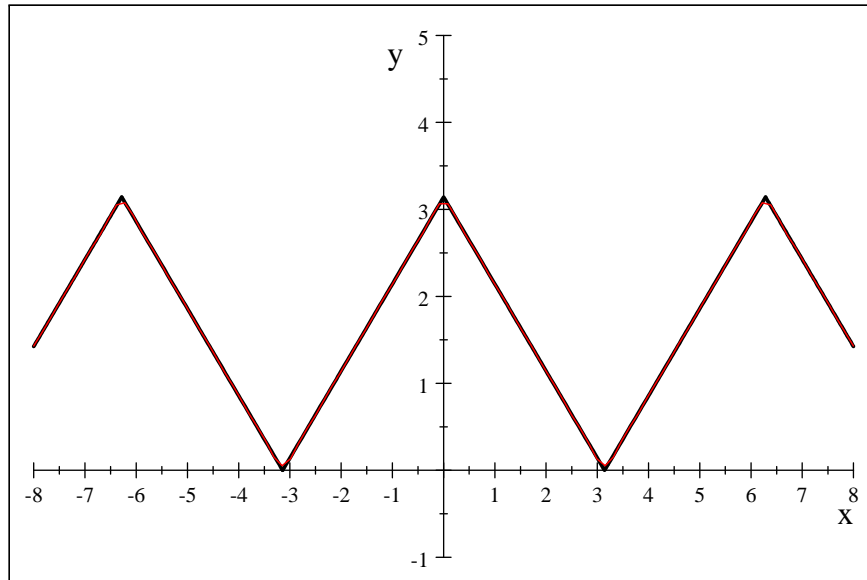


Figure 10: Plot of the triangular wave and $S_5(x)$

3. From $h(x) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}$, if we let $x = 0$, we

get

$$\begin{aligned}\pi &= \frac{\pi}{2} + \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ \frac{\pi}{2} &= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ \frac{\pi^2}{8} &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \\ &= 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots\end{aligned}$$

This allows us to approximate π^2 .

4 Fourier Series of Functions with Arbitrary Periods

Theorem 25 *Suppose that f is a $2p$ -periodic piecewise smooth function. The Fourier series of f is given by*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right] \quad (13)$$

where

$$a_0 = \frac{1}{2p} \int_{-p}^p f(x) dx \quad (14)$$

$$a_n = \frac{1}{p} \int_{-p}^p f(x) \cos \frac{n\pi x}{p} dx \quad (15)$$

and

$$b_n = \frac{1}{p} \int_{-p}^p f(x) \sin \frac{n\pi x}{p} dx \quad (16)$$

The Fourier series converges to $f(x)$ if f is continuous at x and to $\frac{f(x-) + f(x+)}{2}$ otherwise.

We finish this section by noticing that in the special cases that f is either even or odd, the series simplifies greatly.

Theorem 26 *Suppose that f is $2p$ -periodic and has the Fourier series representation*

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right]$$

Then:

1. f is even if and only if $b_n = 0$ for all n and in this case

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{p}$$

2. f is odd if and only if $a_n = 0$ for all n and in this case

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{p}$$

5 Some Applications

In the examples, we saw how we could use Fourier series to approximate π . One of the main uses of Fourier series is in solving some of the differential equations from mathematical physics such as the wave equation or the heat equation. Fourier developed his theory by working on the heat equation. Fourier series also have applications in music synthesis and image processing. All these will be presented in another talk. We will mention the relationship between sound (music) and Fourier series.

When we represent a signal $f(t)$ by its Fourier series $f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$, we are finding the contribution of each frequency $\frac{n\pi}{p}$ to the signal. The value of the corresponding coefficients give us that contribution. The n^{th} term of the partial sum of the Fourier series, $a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p}$, is called the n^{th}

harmonic of f . Its **amplitude** is given by $\sqrt{a_n^2 + b_n^2}$. Conversely, we can create a signal by using the Fourier series $a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right]$ for a given value of p and playing with the value of the coefficients.

Audio signals describe air pressure variations captured by our ears and perceived as sounds. We will focus here on periodic audio signals also known as **tones**. Such signals can be represented by Fourier series.

A **pure tone** can be written as $x(t) = a \cos(\omega t + \phi)$ where $a > 0$ is the amplitude, $\omega > 0$ is the frequency in radians/seconds and ϕ is the phase angle. An alternative way to represent the frequency is in Hertz. The frequency f in Hertz is given by $f = \frac{\omega}{2\pi}$.

The **pitch** of a pure tone is logarithmically related to the frequency. An **octave** is a frequency range between f and $2f$ for a given frequency f in Hertz. Tones separated by an octave are perceived by our ears to be very similar. In

western music, an octave is divided into 12 **notes** equally spaced on the logarithmic scale. The ordering of notes in the octave beginning at the frequency 220 Hz are shown below

Note	A	A#	B	C	C#	D	D#	E
Frequency (Hz)	220	233	247	262	277	294	311	330

A more complicated tone can be represented by a Fourier series of the form

$$x(t) = a_1 \cos(\omega t + \phi_1) + a_2 \cos(\omega t + \phi_2) + \dots$$

Some sites related to sound and music:

- <http://www.jhu.edu/~signals/listen-new/listen-newindex.htm>
- <http://www.jhu.edu/~signals/listen/music1.html>

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