4.4 Monotone Sequences and Cauchy Sequences

4.4.1 Monotone Sequences

The techniques we have studied so far require we know the limit of a sequence in order to prove the sequence converges. However, it is not always possible to find the limit of a sequence by using the definition, or the limit rules. This happens when the formula defining the sequence is too complex to work with. It also happens with sequences defined recursively. Furthermore, it is often the case that it is more important to know if a sequence converges than what it converges to. In this section, we look at two ways to prove a sequence converges without knowing its limit.

We begin by looking at sequences which are monotone and bounded. These terms were defined at the beginning of this chapter.

You will recall that in order to show that a sequence is increasing, several methods can be used.

1. Direct approach, simply show that $a_{n+1} \geq a_n$ for every n.
2. Equivalently, show that $a_{n+1} - a_n \geq 0$ for every n.
3. Equivalently, show that $\dfrac{a_{n+1}}{a_n} \geq 1$ for every n if both a_n and a_{n+1} are positive.
4. If $a_n = f(n)$, one can show a sequence (a_n) is increasing by showing that f is increasing that is by showing that $f'(x) \geq 0$.
5. By induction.

We now state and prove an important theorem about the convergence of increasing sequences.

Theorem 315 *An increasing sequence (a_n) which is bounded above converges. Furthermore, $\lim\limits_{n\to\infty} a_n = \sup\{a_n\}$.*

Proof. *We need to show that given $\epsilon > 0$, there exists N such that $n \geq N \Longrightarrow |a_n - A| < \epsilon$ where A is the limit. Let $\epsilon > 0$ be given. Since (a_n) is bounded above, by theorem 141, $\{a_n\}$ has a supremum. Let $A = \sup\{a_n\}$. Let $\epsilon > 0$ be given. then, $A - \epsilon < A$. By theorem 135, there exists an element of $\{a_n\}$, call it a_N, such that $A - \epsilon < a_N \leq A$. Since (a_n) is increasing, we have $a_n \geq a_N$ for every $n \geq N$. Therefore, if $n \geq N$,*

$$\begin{aligned} A - \epsilon < a_n \leq A &\Longleftrightarrow -\epsilon < a_n - A \leq 0 \\ &\Longleftrightarrow 0 \leq A - a_n < \epsilon \\ &\Longrightarrow |a_n - A| < \epsilon \end{aligned}$$

■

Corollary 316 *A decreasing sequence (a_n) which is bounded below converges. Furthermore, $\lim\limits_{n\to\infty} a_n = \inf\{a_n\}$.*

Proof. *The proof is similar to the proof of the theorem.* ■

Theorem 317 *A monotone sequence converges if and only if it is bounded.*

Proof. *The proof follows from results proven in the previous section and the previous theorem and its corollary. Suppose we have a monotone sequence (x_n). If we assume that (x_n) converges, then it follows that it is bounded by theorem 288 that it is bounded. Conversely, if we assume that (x_n) is bounded then it is both bounded above and below. Since (x_n) is monotone, then it is either increasing or decreasing. If it is increasing, it will converge by the previous theorem since it is also bounded above. If it is decreasing, then it will converge by the corollary since it is bounded below.* ■

Example 318 *Prove that the sequence whose general term is $a_n = \sum\limits_{k=0}^{n} \frac{1}{k!}$ converges.*

We try to establish this result by showing that this sequence is non-decreasing and bounded above.

- (a_n) *is increasing:* $a_n = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + ... + \frac{1}{n!}$. *Therefore,*

$$a_{n+1} - a_n = \frac{1}{(n+1)!} > 0$$

- (a_n) *is bounded above. For this, we use the fact that $n! \geq 2^{n-1}$ for every $n \geq 1$. The proof of this fact is left as an exercise. Therefore,*

$$\begin{aligned}
1 + 1 + \frac{1}{2} + \frac{1}{3!} + ... + \frac{1}{n!} &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} \\
&< 1 + \left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + ... + \left(\frac{1}{2}\right)^{n-1} \\
&< 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} \\
&< 1 + \frac{1}{1 - \frac{1}{2}} \\
&< 3
\end{aligned}$$

- *Since (a_n) is bounded above and increasing, it must converge. We will call the limit e'.*

Example 319 *Find* $\lim\limits_{n\to\infty} a_n$ *where* (a_n) *is defined by:*

$$\begin{aligned} a_1 &= 2 \\ a_{n+1} &= \frac{1}{2}(a_n + 6) \end{aligned}$$

If we knew the limit existed, finding it would be easy. We must first establish that it exists. We do this by showing that this sequence is increasing and bounded above. This part is left as an exercise. Once this fact has been established, then we know the sequence must converge. Let L *be its limit, we must find* L*. Before proceeding, we will recall the following fact:* $\lim a_{n+1} = \lim a_n$*. Therefore, if for every* n *we have* $a_{n+1} = \frac{1}{2}(a_n + 6)$*, then we must also have:*

$$\begin{aligned} \lim a_{n+1} &= \lim \frac{1}{2}(a_n + 6) \\ L &= \frac{1}{2}(\lim a_n + 6) \\ L &= \frac{1}{2}(L + 6) \\ 2L &= L + 6 \\ L &= 6 \end{aligned}$$

Example 320 *Let* $a_n = \left(1 + \frac{1}{n}\right)^n$*. Show* $\lim a_n$ *exists. This limit is in fact the number e, but we won't show that. Again, to show that* (a_n) *converges, we show that it is increasing and bounded above.*

- (a_n) *bounded above. To establish this, we use the following fact: If* k *is an integer such that* $1 < k \leq n$*, then*

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right)$$

  The proof of this is left as an exercise. Using the binomial theorem, we have:

$$\begin{aligned} a_n &= 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \ldots + \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}\left(\frac{1}{n}\right)^k + \ldots + \frac{n!}{n!}\left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \ldots + \frac{1}{k!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right) + \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n} \right. \\ &< 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!} \\ &< e' \end{aligned}$$

  where e' *is the limit found in the exercise above. Thus,* (a_n) *is bounded by* e'*.*

- *(a_n) increasing. The kth term in the expansion of a_n is $\frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{k-1}{n}\right)$.*
  The kth term in the expansion of a_{n+1} is $\frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)...\left(1-\frac{k-1}{n+1}\right)$.
  Since $\frac{j}{n+1}<\frac{j}{n}$, it follows that $1-\frac{j}{n+1}>1-\frac{j}{n}$. Hence, $a_{n+1}>a_n$.
  So, (a_n) is increasing.
- *Since (a_n) is increasing and bounded above, it follows that (a_n) converges. We define this limit to be the number e.*

4.4.2 Cauchy Sequences

Definition 321 (Cauchy Sequence) *A sequence (x_n) is said to be a Cauchy sequence if for each $\epsilon>0$ there exists a positive integer N such that $m,n\geq N\Longrightarrow|x_m-x_n|<\epsilon$.*

We begin with some remarks.

Remark 322 *These series are named after the French mathematician Augustin Louis Cauchy (1789-1857).*

Remark 323 *It is important to note that the inequality $|x_m-x_n|<\epsilon$ must be valid for all integers m,n that satisfy $m,n\geq N$. In particular, a sequence (x_n) satisfying $|x_{n+1}-x_n|<\epsilon$ for all $n\geq N$ may not be a Cauchy sequence.*

Remark 324 *A Cauchy sequence is a sequence for which the terms are eventually close to each other.*

Remark 325 *In theorem 289, we proved that if a sequence converged then it had to be a Cauchy sequence. In fact, as the next theorem will show, there is a stronger result for sequences of real numbers.*

We now look at some examples.

Example 326 *Consider (x_n) where $x_n=\frac{1}{n}$. Prove that this is a Cauchy sequence.*
Let $\epsilon>0$ be given. We want to show that there exists an integer $N>0$ such that $m,n<N\Longrightarrow|x_m-x_n|<\epsilon$. That it we would like to have

$$|x_n-x_m|<\epsilon\Longleftrightarrow\left|\frac{1}{n}-\frac{1}{m}\right|<\epsilon$$

Since

$$\left|\frac{1}{n}-\frac{1}{m}\right|<\frac{1}{n}+\frac{1}{m}$$

If we make $\frac{1}{n}+\frac{1}{m}<\epsilon$, the result will follow. This will happen if both $\frac{1}{n}<\frac{\epsilon}{2}$ that is when $n>\frac{2}{\epsilon}$ and $\frac{1}{m}<\frac{\epsilon}{2}$ that is when $n>\frac{2}{\epsilon}$. So, we see that if N is an integer larger than $\frac{2}{\epsilon}$ then $m,n>N\Longrightarrow|x_m-x_n|<\epsilon$.

Example 327 *Consider* (x_n) *where* $x_n = \sum_{k=1}^{n} \frac{1}{k^2}$
Let $\epsilon > 0$ *be given. We want to show that there exists an integer* $N > 0$ *such that* $m, n < N \Longrightarrow |x_m - x_n| < \epsilon$ *(without loss of generality, let us assume that* $n > m$*). That it we would like to have*

$$\begin{aligned} |x_n - x_m| &< \epsilon \Longleftrightarrow \left| \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{m} \frac{1}{k^2} \right| < \epsilon \\ &\Longleftrightarrow \left| \sum_{k=m+1}^{n} \frac{1}{k^2} \right| < \epsilon \end{aligned}$$

Remembering telescoping sums and the fact that $\frac{1}{k^2} < \frac{1}{k(k-1)}$*, we see that*

$$\begin{aligned} |x_n - x_m| &= \left| \sum_{k=m+1}^{n} \frac{1}{k^2} \right| \\ &= \sum_{k=m+1}^{n} \frac{1}{k^2} \\ &< \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\ &= \frac{1}{m} - \frac{1}{n} \\ &< \frac{1}{m} + \frac{1}{n} \end{aligned}$$

So, as we saw in the previous example, if N *is an integer larger than* $\frac{\epsilon}{2}$*, then* $m, n < N \Longrightarrow |x_m - x_n| < \epsilon$*.*

We now look at important properties of Cauchy sequences.

Theorem 328 *Every Cauchy sequence is bounded.*
Proof. *See problems.* ■

Theorem 329 *A sequence of real numbers converges if and only if it is a Cauchy sequence.*
Proof. *We have already proven one direction. Let* (x_n) *be a sequence of real numbers.*

- *If* (x_n) *converges, then we know it is a Cauchy sequence by theorem 289.*
- *Assume* (x_n) *is a Cauchy sequence. We must prove that it converges. By theorem 328, we know that* $\{x_n\}$ *is bounded. Therefore, by the completeness axiom, for each* n*, the number*

$$a_n = \inf \{x_n, x_{n+1}, x_{n+2}, ...\}$$

exists. Furthermore, the sequence (a_n) is increasing and bounded (why?). Therefore, by theorem 317, (a_n) converges. Let $a = \lim a_n$. We prove that $\lim x_n = a$. Let $\epsilon > 0$ be given. Since (x_n) is a Cauchy sequence, we can find $N > 0$ such that $m, n \geq N \Longrightarrow |x_n - x_m| < \frac{\epsilon}{2}$. It follows that both (x_n) and (a_n) are contained in the interval $\left[x_N - \frac{\epsilon}{2}, x_N + \frac{\epsilon}{2}\right]$ (why). Thus, the limit a is also in this interval. Therefore, for all $n \geq N$, we have:

$$\begin{aligned} |x_n - a| &\leq |x_n - x_N| + |x_N - a| \\ &< \epsilon \end{aligned}$$

This proves that $\lim x_n = a$, thus (x_n) converges.

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Remark 330 *The key to this theorem is that we are dealing with a sequence of real numbers. The fact that a Cauchy sequence of real number converges is linked to the fact that $\mathbb{R}$ is complete. In fact, this is sometimes used as a definition of completeness. Some texts say that a set is complete if every Cauchy sequence converges in that set. It is possible to find a Cauchy sequence of rational numbers which does not converge in $\mathbb{Q}$.*

We finish this section with an important theorem.

Definition 331 *A* ***nested sequence of intervals*** *is a sequence $\{I_n\}$ of intervals with the property that $I_{n+1} \subseteq I_n$ for all n.*

Theorem 332 (Nested Intervals) *If $\{[a_n, b_n]\}$ is a nested sequence of closed intervals then there exists a point z that belongs to all the intervals. Furthermore, if $\lim a_n = \lim b_n$ then the point z is unique.*

Proof. *We provide a sketch of the proof and leave the details as homework. First, prove that $\{a_n\}$ and $\{b_n\}$ are monotone and bounded. Thus, they must converge. Let $a = \lim a_n$ and $b = \lim b_n$. Then, explain why $a \leq b$. Conclude from this.* ■

4.4.3 Exercises

1. Prove that $n! \geq 2^{n-1}$ for every $n \geq 1$

2. Prove that if k is an integer such that $1 < k \leq n$, then
$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{n^k} = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{k-1}{n}\right)$$

3. Prove that the sequence given by
$$\begin{aligned} a_1 &= 2 \\ a_{n+1} &= \frac{1}{2}(a_n + 6) \end{aligned}$$
is increasing and bounded above by 6. (hint: use induction for both).

4. Show that the sequence defined by

$$a_1 = 1$$
$$a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and satisfies $a_n < 3$ for all n. Then, find its limit.

5. Let a and b be two positive numbers such that $a > b$. Let a_1 be their arithmetic mean, that is $a_1 = \frac{a+b}{2}$. Let b_1 be their geometric mean, that is $b_1 = \sqrt{ab}$. Define $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$.

    (a) Use mathematical induction to show that $a_n > a_{n+1} > b_{n+1} > b_n$.

    (b) Deduce that both (a_n) and (b_n) converge.

    (c) Show that $\lim a_n = \lim b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean**.

6. Prove theorem 328.

7. Answer the why? parts in the proof of theorem 329.

8. Finish proving theorem 332.