## 3.3 Limit Points

### 3.3.1 Main Definitions

Intuitively speaking, a limit point of a set $S$ in a space $X$ is a point of $X$ which can be approximated by points of $S$ other than $x$ as well as one pleases.

The notion of limit point is an extension of the notion of being "close" to a set in the sense that it tries to measure how crowded the set is. To be a limit point of a set, a point must be surrounded by an infinite number of points of the set.

We now give a precise mathematical definition. In what follows, $\mathbb{R}$ is the reference space, that is all the sets are subsets of $\mathbb{R}$.

**Definition 240 (Limit point)** *Let $S \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$.*

1. *$x$ is a **limit point** or an **accumulation point** or a **cluster point** of $S$ if $\forall \delta > 0$, $(x-\delta, x+\delta) \cap S \setminus \{x\} \neq \varnothing$.*

2. *The set of limit points of a set $S$ is denoted $L(S)$*

**Remark 241** *Let us remark the following:*

1. *In the above definition, we can replace $(x-\delta, x+\delta)$ by a neighborhood of $x$. Therefore, $x$ is a limit point of $S$ if any neighborhood of $x$ contains points of $S$ other than $x$.*

2. *Being a limit point of a set $S$ is a stronger condition than being close to a set $S$. It requires any neighborhood of the limit point $x$ to contain points of $S$ other than $x$.*

3. *The above two remarks should make it clear that $L(S) \subseteq \overline{S}$. We can also prove it rigorously. If $x \in L(S)$ then $\exists \delta > 0 : (x-\delta, x+\delta) \cap S \setminus \{x\} \neq \varnothing$. Let $y \in (x-\delta, x+\delta) \cap S \setminus \{x\}$. Then, $y \in (x-\delta, x+\delta)$ and $y \in S \setminus \{x\}$. So, $y \in S$ thus $(x-\delta, x+\delta) \cap S \neq \varnothing$. Hence $x \in \overline{S}$.*

Let us first look at easy examples to understand what a limit point is and what the set of limit points of a given set might look like.

**Example 242** *Let $S = (a, b)$ and $x \in (a, b)$. Then $x$ is a limit point of $(a, b)$. Let $\delta > 0$ and consider $(x-\delta, x+\delta)$. This interval will contain points of $(a, b)$ other than $x$, infinitely many points in fact.*

**Example 243** *$a$ and $b$ are limit points of $(a, b)$. Given $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains infinitely many points of $(a, b) \setminus \{a\}$ thus showing $a$ is a limit point of $(a, b)$. The same is true for $b$.*

**Example 244** *Let $S = [a, b]$ and $x \in [a, b]$. Then $x$ is a limit point of $[a, b]$. This is true for the same reasons as above.*

At this point you may think that there is no difference between a limit point and a point close to a set. Consider the next example.

**Example 245** *Let* $S = (0,1) \cup \{2\}$*. 2 is close to* $S$*. For any* $\delta > 0$*,* $\{2\} \subseteq (2-\delta, 2+\delta) \cap S$ *so that* $(2-\delta, 2+\delta) \cap S \neq \varnothing$*. But 2 is not a limit point of* $S$*.* $(2-.1, 2+.1) \cap S \setminus \{2\} = \varnothing$*.*

**Remark 246** *You can think of a limit point as a point close to a set but also surrounded by many (infinitely many) points of the set.*

We look at more examples.

**Example 247** $L((a,b)) = [a,b]$*. From the first two examples of this section, we know that* $L((a,b)) \subseteq [a,b]$*. What is left to show is that if* $x \notin [a,b]$ *then* $x$ *is not a limit point of* $(a,b)$*. Let* $x \in \mathbb{R}$ *such that* $x \notin (a,b)$*. Assume* $x > b$ *(the case* $a < a$ *is similar and left for the reader to check). Let* $\delta < |x-b|$*. Then* $(x-\delta, x+\delta) \cap (a,b) \setminus \{x\} = \varnothing$*.*

**Example 248** $L(\mathbb{Z}) = \varnothing$*. Let* $x \in \mathbb{R}$*. For* $x$ *to be a limit point, we must have that for any* $\delta > 0$ *the interval* $(x-\delta, x+\delta)$ *would have to contain points of* $\mathbb{Z}$ *other than* $x$*. Since for every real number* $x$ *there exists an integer* $n$ *such that* $n-1 \leq x < n$*, if* $\delta < \min(|x-n+1|, |x-n|)$*, the interval* $(x-\delta, x+\delta)$ *contains no integer therefore* $(x-\delta, x+\delta) \cap \mathbb{Z} \setminus \{x\} = \varnothing$*. So, no real number can be a limit point of* $\mathbb{Z}$*.*

**Example 249** $L(\mathbb{Q}) = \mathbb{R}$*. For any real number and for any* $\delta > 0$*, the interval* $(x-\delta, x+\delta)$ *contains infinitely many points of* $\mathbb{Q}$ *other than* $x$*. Thus,* $(x-\delta, x+\delta) \cap \mathbb{Q} \setminus \{x\} \neq \varnothing$*. So every real number is a limit point of* $\mathbb{Q}$*.*

**Example 250** *If* $S = (0,1) \cup \{2\}$ *then* $\bar{S} = [0,1] \cup \{2\}$ *but* $L(S) = [0,1]$*. Using arguments similar to the arguments used in the previous examples, one can prove that every element in* $[a,b]$ *is a limit point of* $S$ *and every element outside of* $[a,b]$ *is not. You will notice that* $2 \in S$ *but* $2 \notin L(S)$*. Also,* $0$ *and* $1$ *are in* $L(S)$ *but not in* $S$*. So, in the most general case, one cannot say anything regarding whether* $S \subseteq L(S)$ *or* $L(S) \subseteq S$*.*

**Example 251** *If* $S = \left\{ \dfrac{1}{n} : n \in \mathbb{Z}^+ \right\}$ *then* $L(S) = \{0\}$ *(see problems).*

### 3.3.2 Properties

**Theorem 252** *Let* $x \in \mathbb{R}$ *and* $S \subseteq \mathbb{R}$*.*

1. *If* $x$ *has a neighborhood which only contains finitely many members of* $S$ *then* $x$ *cannot be a limit point of* $S$*.*
2. *If* $x$ *is a limit point of* $S$ *then any neighborhood of* $x$ *contains infinitely many members of* $S$*.*

***Proof.*** *We prove each part separately.*

**Part 2:** *This part follow immediately from part 1.*

**Part 1:** *Let $U$ be a neighborhood of $x$ which contains only a finite number of points of $S$ that is $U \cap S$ is finite. Then, $U \cap S \setminus \{x\}$ is also finite. Suppose $U \cap S \setminus \{x\} = \{y_1, y_2, ..., y_n\}$. We show there exists $\delta > 0$ such that $(x - \delta, x + \delta) \cap U$ does not contain any member of $S \setminus \{x\}$. Since both $(x - \delta, x + \delta)$ and $U$ are neighborhood of $x$, so is their intersection. This will prove there is a neighborhood of $x$ containing no element of $S \setminus \{x\}$ hence proving $x$ is not a limit point of $S$. Let $\delta = \min\{|x - y_1|, |x - y_2|, ..., |x - y_n|\}$. Since $x$ is not equal to $y_i$, $\delta > 0$. Then $(x - \delta, x + \delta) \cap U$ Contains no points of $S$ other than $x$ thus proving our claim.*

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***Corollary 253*** *No finite set can have a limit point.*
***Proof.*** *Follows immediately from the theorem.* ■

The next theorem is very important. It helps understand the relationship between the set of limit points of a set and the closure of a set.

**Theorem 254** *Let $S \subseteq \mathbb{R}$*

$$\bar{S} = S \cup L(S)$$

***Proof.*** *We show inclusion both ways.*

1. $S \cup L(S) \subseteq \bar{S}$
   *We already know that $S \subseteq \bar{S}$. We now show that $L(S) \subseteq \bar{S}$. This will imply the result. Suppose that $x \in L(S)$. Then $\forall \delta > 0$, $(x - \delta, x + \delta) \cap S \setminus \{x\} \neq \varnothing$. It follows that $(x - \delta, x + \delta) \cap S \neq \varnothing$, since $S \setminus \{x\} \subseteq S$. Therefore, $x$ is in the closure of $S$.*

2. $\bar{S} \subseteq S \cup L(S)$
   *Let $x \in \bar{S}$. We need to prove that $x \in S \cup L(S)$. Either $x \in S$ or $x \notin S$. If $x \in S$ then $x \in S \cup L(S)$. If $x \notin S$ then $x$ is close to $S$. Therefore, $\forall \delta > 0$, $(x - \delta, x + \delta) \cap S \neq \varnothing$. Since $x \notin S$, $S = S \setminus \{x\}$, therefore $(x - \delta, x + \delta) \cap S \setminus \{x\} \neq \varnothing$. It follows that $x \in L(S)$ and therefore $x \in S \cup L(S)$. So we see that in all cases, if we assume that $x \in \bar{S}$ then we must have $x \in S \cup L(S)$.*

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### 3.3.3 Important Facts to Remember

- Definitions properties and theorems in this section.
- $L(A) \cup L(B) = L(A \cup B)$ (see problems)

- $\bar{S} = S \cup L(S)$
- $L(S) \subseteq \bar{S}$
- $L(S)$ is closed (see problems).
- If $U$ is open then $L(U) = \bar{U}$ (see problems).

### 3.3.4 Exercises

1. Prove that if $S = \left\{ \frac{1}{n} : n \in \mathbb{Z}^+ \right\}$ then $L(S) = \{0\}$

2. In each situation below, give an example of a set which satisfies the given condition.

   (a) An infinite set with no limit point.
   (b) A bounded set with no limit point.
   (c) An unbounded set with no limit point.
   (d) An unbounded set with exactly one limit point.
   (e) An unbounded set with exactly two limit points.

3. Prove that if $A$ and $B$ are subsets of $\mathbb{R}$ and $A \subseteq B$ then $L(A) \subseteq L(B)$.

4. Prove that if $A$ and $B$ are subsets of $\mathbb{R}$ then $L(A) \cup L(B) = L(A \cup B)$.

5. Is it true that if $A$ and $B$ are subsets of $\mathbb{R}$ then $L(A) \cap L(B) = L(A \cap B)$? Give an answer in the following cases:

   (a) $A$ and $B$ are closed.
   (b) $A$ and $B$ are open.
   (c) $A$ and $B$ are intervals.
   (d) General case.

6. Given that $S \subseteq \mathbb{R}$, $S \neq \varnothing$ and $S$ bounded above but $\max S$ does not exist, prove that $\sup S$ must be a limit point of $S$. State and prove a similar result for $\inf S$.

7. Let $S \subseteq \mathbb{R}$. Prove that $L(S)$ must be closed.

8. Prove that if $U$ is open then $L(U) = \bar{U}$.