

5.2 Line Integrals

5.2.1 Introduction

Let us quickly review the kind of integrals we have studied so far before we introduce a new one.

1. Definite integral. Given a continuous real-valued function f , $\int_a^b f(x) dx$ represents the area below the graph of f , between $x = a$ and $x = b$, assuming that $f(x) \geq 0$ between $x = a$ and $x = b$.
2. The definite integral can also be used to compute the length of a curve. If a curve C is given by its position vector $\vec{r}(t) = \langle x(t), y(t) \rangle$ in 2-D or $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ in 3-D for $a \leq t \leq b$, then the length L of the curve C is given by

$$L = \int_a^b |\vec{r}'(u)| du$$

The arc length function was defined to be

$$s(t) = \int_a^t |\vec{r}'(u)| du$$

so that, using the fundamental theorem of Calculus, we have

$$\frac{ds}{dt} = |\vec{r}'(t)|$$

3. Double integrals. Given a real-valued function f of two variables, $\iint_D f(x, y) dA$ represents the volume of the solid above the region D and below the graph of $z = f(x, y)$.
4. The double integral can also be used to find the area of a region by the formula

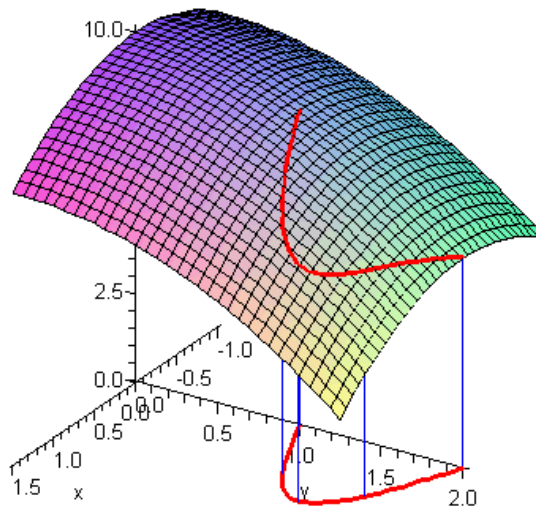
$$\text{area of } D = \iint_D dA$$

In this section, we study an integral similar to the one in example 1, except that instead of integrating over an interval, we integrate along a curve.

5.2.2 Line Integrals Along Plane Curves

Let us consider the following problem: Suppose that we have a plane curve C given by its position vector

$$\vec{r}(t) = \langle x(t), y(t) \rangle \text{ for } a \leq t \leq b. \quad (5.1)$$



Let us assume C is a smooth curve (\vec{r}' is continuous and $\vec{r}'(t) \neq \vec{0}$). Suppose further that we have a continuous function $z = f(x, y)$, we will assume for now $f(x, y) \geq 0$. Consider the surface S given by $\langle x(t), y(t), z \rangle$. This surface will intersect the graph of $z = f(x, y)$ in a curve C' . We wish to find the surface area of S between the curves C and C' . To help you visualize S , think of a curtain hanging. Except that the curtain is not hanging along a straight rail, but a curved one. Furthermore, the rail is not necessarily horizontal, it has whatever shape f has. This is shown in figure 5.2.2. Imagine we want to find the area of the curtain. We will first approximate the area using a technique similar to the one used when defining the definite integral. We will outline the steps.

1. Split $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal length. Let $x_i = x(t_i)$, $y_i = y(t_i)$.
2. The corresponding points $P_i(x_i, y_i)$ divide C into n subarcs of length $\Delta s_1, \Delta s_2, \dots, \Delta s_n$.
3. In each subarc, pick a point $P_i^*(x_i^*, y_i^*)$ (this corresponds to a point t_i^* in $[t_{i-1}, t_i]$).
4. Draw the rectangle with base $[P_{i-1}, P_i]$ and height $f(x_i^*, y_i^*)$. The area of this rectangle is $f(x_i^*, y_i^*) \Delta s_i$.
5. The area of S can be approximated by $\sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$.
6. The larger n is, the better the approximation.

This allows us to define:

Definition 388 With the notation above, the area of S , denoted $A(S)$ is defined to be

$$A(S) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i$$

Definition 389 If f is any continuous function (not just a positive one), defined on a smooth curve C given in equation 5.1, then the **line integral of f along C** is defined by

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i \quad (5.2)$$

if this limit exists.

You will note that we are integrating with respect to arc length. Remembering that $\frac{ds}{dt} = |\vec{r}'(t)|$, it follows that $ds = |\vec{r}'(t)| dt$ and therefore, the line integral can be evaluated as follows:

Theorem 390 If f is any continuous function (not just a positive one), defined on a smooth curve C given in equation 5.1, then the **line integral of f along C** can be computed by the following formula

$$\begin{aligned} \int_C f(x, y) ds &= \int_a^b f(x(t), y(t)) |\vec{r}'(t)| dt \\ &= \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \end{aligned} \quad (5.3)$$

Remark 391 We used a and b for the limits of integration because they are the limits of the variable t .

Remark 392 Note that the line integral is with respect to arc length. However, to compute it, we use the parametrization of the curve, whatever it is. We rewrite everything in terms of the parameter used for the curve.

Example 393 Evaluate $\int_C (2 + x^2y) ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$.

First, we must write C in parametric form. The upper half of the unit circle is

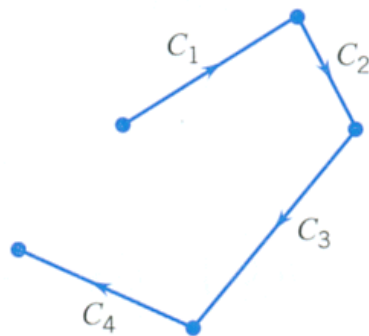


Figure 5.5: Piecewise smooth curve

$x(t) = \cos t, y(t) = \sin t, 0 \leq t \leq \pi$. Then

$$\begin{aligned}
 \int_C f(x, y) ds &= \int_a^b f(x(t), y(t)) |\vec{r}'(t)| dt \\
 &= \int_0^\pi (2 + \cos^2 t \sin t) \sqrt{\sin^2 t + \cos^2 t} dt \\
 &= \int_0^\pi (2 + \cos^2 t \sin t) dt \\
 &= \int_0^\pi 2 dt + \int_0^\pi \cos^2 t \sin t dt \\
 &= \left[2t - \frac{\cos^3 t}{3} \right]_0^\pi \\
 &= 2\pi + \frac{1}{3} + \frac{1}{3} \\
 &= 2 \left(\pi + \frac{1}{3} \right)
 \end{aligned}$$

Remark 394 In the above theorem, the given formula to find $\int_C f(x, y) ds$ requires that C be a smooth curve. However, It is still possible to compute a line integral when the curve C is not a smooth curve, as long as it is piecewise smooth, that is made of smooth pieces, as the one shown in figure 5.5. In this case

$$\int_C f(x, y) ds = \int_{C_1} f(x, y) ds + \int_{C_2} f(x, y) ds + \int_{C_3} f(x, y) ds + \int_{C_4} f(x, y) ds$$

Example 395 Evaluate $\int_C 2x ds$ where $C = C_1 \cup C_2$, C_1 being the arc of the parabola $y = x^2$ between $(0, 0)$ and $(1, 1)$ and C_2 being the vertical line from $(1, 1)$ to $(1, 2)$.

$$\int_C 2x ds = \int_{C_1} 2x ds + \int_{C_2} 2x ds$$

We evaluate each integral separately.

1. To evaluate $\int_{C_1} 2x ds$, we need to parametrize C_1 . $y = x^2$ can be parametrized by $x = t$, $y = t^2$, $0 \leq t \leq 1$. Thus

$$\begin{aligned} \int_{C_1} 2x ds &= \int_0^1 2t \sqrt{1 + 4t^2} dt \\ &= \frac{5\sqrt{5} - 1}{6} \end{aligned}$$

2. To evaluate $\int_{C_2} 2x ds$, we need to parametrize C_2 . A vertical line between the given points can be parametrized by $x = 1$, $y = t$, $1 \leq t \leq 2$. Thus

$$\begin{aligned} \int_{C_2} 2x ds &= \int_1^2 2\sqrt{1} dt \\ &= 2 \end{aligned}$$

3. Therefore

$$\int_C 2x ds = \frac{5\sqrt{5} - 1}{6} + 2$$

Two other integrals can be obtained using a similar technique. When we form the sum, we can use $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$ instead of Δs_i . The integrals we obtain are

$$\int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i \quad (5.4)$$

$$\int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i \quad (5.5)$$

These are called line integrals of f along C with respect to x and y . If $x = x(t)$, then $dx = x'(t) dt$. Similarly, $dy = y'(t) dt$. So, these integrals can be computed as follows:

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt \quad (5.6)$$

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

Remark 396 It often happens that these integrals appear together as in $\int_C P(x, y) dx + \int_C Q(x, y) dy$. In this case, we will write

$$\int_C P(x, y) dx + \int_C Q(x, y) dy = \int_C P(x, y) dx + Q(x, y) dy$$

Remark 397 The line integral in equation 5.3 is called the line integral of f along C with respect to **arc length**. The line integrals in equation 5.6 are called line integrals of f along C with respect to x and y .

Remark 398 As you have noticed, to evaluate a line integral, one has to first parametrize the curve over which we are integrating. Here are some pointers on how to do it.

1. **Circle of radius r :**

- **Counter-clockwise:** $x = r \cos t$, $y = r \sin t$ with $0 \leq t \leq 2\pi$.
- **Clockwise:** $x = r \cos t$, $y = -r \sin t$ with $0 \leq t \leq 2\pi$.

2. **A curve given by a function $y = f(x)$:** $x = t$, $y = f(t)$. For example, $y = x^2$ can be parametrized by $x = t$, $y = t^2$.

3. **Vertical line through (a, b) :** $x = a$, $y = t$.

4. **Horizontal line through (a, b) :** $x = t$, $y = b$.

5. **Line segment between $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{r}_1 = \langle x_1, y_1, z_1 \rangle$:**

- **Vector form:** $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \leq t \leq 1$.
- **In coordinate form:** $x(t) = (1-t)x_0 + tx_1$, $y(t) = (1-t)y_0 + ty_1$,
 $z(t) = (1-t)z_0 + tz_1$.
- Note that it is similar in 2-D, simply drop the z -coordinate.

Example 399 Evaluate $\int_C y^2 dx + x dy$ in each case below:

1. C is the line segment from $(-5, -3)$ to $(0, 2)$.

2. C is the arc of the parabola $x = 4 - y^2$ between $(-5, -3)$ and $(0, 2)$.

Solution of 1 C can be parametrized by $x = -5(1-t) + 0t = 5t - 5$ and $y = -3(1-t) + 2t = 5t - 3$ with $0 \leq t \leq 1$. Therefore

$$\begin{aligned} \int_C y^2 dx + x dy &= \int_0^1 (5t - 3)^2 5 dt + \left(\int_0^1 5t - 5 \right) 5 dt \\ &= 5 \int_0^1 (25t^2 - 25t + 4) dt \\ &= -\frac{5}{6} \end{aligned}$$

Solution of 2 C can be parametrized by $y = t$ and $x = 4 - t^2$ with $-3 \leq t \leq 2$. Therefore

$$\begin{aligned} \int_C y^2 dx + x dy &= \int_{-3}^2 t^2 (-2t) dt + (4 - t^2) dt \\ &= \int_{-3}^2 (-2t^3 - t^2 + 4) dt \\ &= \frac{245}{6} \end{aligned}$$

Remark 400 In these two computations, we were evaluating the same integral, between the same points, along different paths. We got different answers. This indicates that the integral depends on the path chosen. We will elaborate on this in the next section.

5.2.3 Line Integrals Along Space Curves

We can define a similar integral if C is a space curve given by .

Definition 401 Let C be a smooth curve given by $x = x(t)$, $y = y(t)$ and $z = z(t)$, $a \leq t \leq b$. The line integral of $f(x, y, z)$ along C is defined to be

$$\int_C f(x, y, z) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

Theorem 402 The above line integral is evaluated using the formula

$$\begin{aligned} \int_C f(x, y, z) ds &= \int_a^b f(x(t), y(t), z(t)) |\vec{r}'(t)| dt \quad (5.7) \\ &= \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

As before, we can define line integrals with respect to x , y , and z . They are evaluated as follows:

$$\begin{aligned} \int_C f(x, y, z) dx &= \int_a^b f(x(t), y(t), z(t)) x'(t) dt \quad (5.8) \\ \int_C f(x, y, z) dy &= \int_a^b f(x(t), y(t), z(t)) y'(t) dt \\ \int_C f(x, y, z) dz &= \int_a^b f(x(t), y(t), z(t)) z'(t) dt \end{aligned}$$

Example 403 Evaluate $\int_C y \sin z ds$ where C is the helix given by $x = \cos t$,

$y = \sin t$, and $z = t$, $0 \leq t \leq 2\pi$.

$$\begin{aligned} \int_C y \sin z \, ds &= \int_0^{2\pi} \sin^2 t \sqrt{\sin^2 t + \cos^2 t + 1} \, dt \\ &= \sqrt{2} \int_0^{2\pi} \sin^2 t \, dt \\ &= \sqrt{2} \int_0^{2\pi} \frac{1}{2} (1 - \cos 2t) \, dt \\ &= \sqrt{2}\pi \end{aligned}$$

Summary 404 *Let us recapitulate what we learned about line integrals of a function along a curve.*

1. If C is a smooth curve in the plane, then to compute $\int_C f(x, y) \, ds$, we do the following:

(a) Find a smooth parametrization of C , say $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \leq t \leq b$.

(b) The integral is evaluated by the formula $\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \left\| \vec{r}'(t) \right\| dt$

2. If C is a smooth curve in space, then to compute $\int_C f(x, y, z) \, ds$, we do the following:

(a) Find a smooth parametrization of C , say $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$.

(b) The integral is evaluated by the formula $\int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \left\| \vec{r}'(t) \right\| dt$

3. If C is made by joining a finite number of smooth curves end to end in other words $C = C_1 \cup C_2 \cup \dots \cup C_n$ then $\int_{C_1 \cup C_2 \cup \dots \cup C_n} f(x, y) \, ds = \int_{C_1} f(x, y) \, ds + \int_{C_2} f(x, y) \, ds + \dots + \int_{C_n} f(x, y) \, ds$.

4. If C is a plane curve and $f(x, y) \geq 0$ then the geometric meaning of $\int_C f(x, y) \, ds$ is the area of the curtain with base C below the graph of $z = f(x, y)$.

5. Line integrals are also used in physics. An important meaning is the following. The mass of a thin wire lying along a smooth curve C is $\int_C f(x, y, z) \, ds$ where $f(x, y, z)$ is the density of the wire at (x, y, z) . This formula allows us to find the mass of a thin wire for which the density is not constant along the wire.

5.2.4 Line Integrals of Vector Fields

In the previous section, we learned the meaning of and how to compute line integrals of scalar functions. We now turn to line integrals of a vector field. We motivate this type of integral with an application.

In physics, work is defined as a force acting upon an object to cause a displacement. There are three key words in this definition - **force**, **displacement**, and **cause**. In order for a force to qualify as having done work on an object, there must be a displacement and the force must cause the displacement. There are several good examples of work which can be observed in everyday life - a horse pulling a plow through the fields, a father pushing a grocery cart down the aisle of a grocery store, a freshman lifting a backpack full of books upon her shoulder, a weight lifter lifting a barbell above her head, an Olympian launching the shot-put, etc. In each case described here there is a force exerted upon an object to cause that object to be displaced.

We first remind the reader of some formulas giving the work done by a force in simple cases. You may recall learning in a physics class that the work W done by a variable force $f(x)$ moving an object from a to b along the x -axis is given by

$$W = \int_a^b f(x) dx \quad (5.9)$$

In this case, we have a variable force, along a straight line, the x -axis. Another simple case is when we have a constant force F which moves an object between two points P and Q in space. The work done in this case is

$$W = F \cdot \overrightarrow{PQ} \quad (5.10)$$

These two cases were fairly simple. In the first one, the motion was along the x -axis. In the second, though the motion was in space, the force was constant. In this section, we wish to compute the work done by a force in a more general setting. The force will be a variable force, the object will be moving along any smooth curve. This can happen when, for example, an object is moving along a curve, in a vector force field. At each point along the curve, the force applied to an object will be given by a vector field. In other words, suppose we have a vector field $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$. We wish to compute the work done by F in moving particles along a smooth curve C .

We will use a technique similar to the technique used in the previous section when we defined line integrals. We will also use the same notation. I will not repeat the details here. C is divided into subarcs $P_{i-1}P_i$ of length Δs_i . In the i^{th} subarc, we select a point $P_i^* = (x_i^*, y_i^*, z_i^*)$. This point corresponds to a value of t we call t_i^* . If Δs_i a particle moves from P_{i-1} to P_i in the direction of $\overrightarrow{T}(t_i^*)$, the unit tangent vector at P_i^* . Thus the work done by F in moving the particle from P_{i-1} to P_i can be approximated by

$$W_i = F(x_i^*, y_i^*, z_i^*) \cdot \Delta s_i \overrightarrow{T}(t_i^*)$$

So, the work done by F moving the particle along C can be approximated by:

$$W = \sum_{i=1}^n F(x_i^*, y_i^*, z_i^*) \cdot \vec{T}(t_i^*) \Delta s_i$$

As n becomes larger, this approximation becomes better. So, we can define:

Definition 405 *The work W done by a force field $F(x, y, z)$ acting on a moving a particle along a smooth curve C can be given by the limit of the above sum as $n \rightarrow \infty$. This is precisely the line integral we defined in the previous section. Thus,*

$$\begin{aligned} W &= \int_C F(x, y, z) \cdot \vec{T}(x, y, z) ds \\ &= \int_C F \cdot \vec{T} ds \end{aligned}$$

You will recall that if the curve C is given by a position vector $\vec{r}(t)$ then

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Also,

$$ds = |\vec{r}'(t)| dt$$

Therefore

$$\begin{aligned} \vec{T} ds &= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)| dt \\ &= \vec{r}'(t) dt \end{aligned}$$

It follows that

$$W = \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

This integral is often abbreviated as $\int_C F \cdot d\vec{r}$. Keep in mind that $F(r(t)) = F(x(t), y(t), z(t))$, also $d\vec{r} = \vec{r}'(t) dt$.

Definition 406 *Let F be a continuous vector field defined on a smooth curve C given by a position vector $\vec{r}(t)$, $a \leq t \leq b$. The line integral of F over C is*

$$\begin{aligned} \int_C F \cdot d\vec{r} &= \int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_C F \cdot \vec{T} ds \end{aligned} \tag{5.11}$$

Remark 407 *Keep in mind that $F(\vec{r}(t)) = F(x(t), y(t), z(t))$, also $d\vec{r} = \vec{r}'(t) dt$. In particular, we must use the parametrized form of C .*

Remark 408 The above formula is valid in both 2-D and 3-D.

Remark 409 There are six different ways to write the integral corresponding to the work of a vector field $F(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ over a curve C given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ for $a \leq t \leq b$. They are shown below. Keep in mind they are different ways of writing the same thing.

- $\int_C F \cdot \vec{T} ds$, the definition.
- $\int_C F \cdot d\vec{r}$, called the compact differential form.
- $\int_a^b F(\vec{r}(t)) \cdot \vec{r}'(t) dt$, since $d\vec{r} = \vec{r}'(t) dt$.
- $\int_a^b F(x(t), y(t), z(t)) \cdot \langle x'(t), y'(t), z'(t) \rangle dt$, using the components of $\vec{r}'(t)$.
- $\int_a^b [P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)] dt$, using the components of F .
- $\int_C Pdx + Qdy + Rdz$, the most common form.

Example 410 Find the work done by $F = \langle y - x^2, z - y^2, x - z^2 \rangle$ over the curve $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ from $(0, 0, 0)$ to $(1, 1, 1)$.

First, let us note that the two points correspond to $t = 0$ and $t = 1$. To evaluate the integral, we proceed as for line integrals of scalar functions. We write everything in terms of t . Now,

$$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$$

Also

$$\begin{aligned} F(x, y, z) &= \langle y - x^2, z - y^2, x - z^2 \rangle \\ &= \langle t^2 - t^2, t^3 - t^4, t - t^6 \rangle \\ &= \langle 0, t^3 - t^4, t - t^6 \rangle \end{aligned}$$

So

$$\begin{aligned} F(\vec{r}(t)) \cdot \vec{r}'(t) &= \langle 0, t^3 - t^4, t - t^6 \rangle \cdot \langle 1, 2t, 3t^2 \rangle \\ &= 2t^4 - 2t^5 + 3t^3 - 3t^8 \end{aligned}$$

Hence

$$\begin{aligned} \text{Work} &= \int_C F \cdot \vec{T} ds \\ &= \int_0^1 F(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8) dt \\ &= \frac{29}{60} \end{aligned}$$

Example 411 An object of mass m moves along the curve given by the position vector $\vec{r}(t) = \langle \alpha t^2, \sin \beta t, \cos \beta t \rangle$, $0 \leq t \leq 1$, α and β are constants. Find the total force acting on the object and the work done by this force. You will recall that Newton's second law of motion says that

$$\begin{aligned} F &= ma(t) \\ &= m\vec{r}''(t) \end{aligned}$$

And therefore, the work W will be given by

$$W = \int_0^1 m\vec{r}''(t) \cdot \vec{r}'(t) dt$$

from equation 5.11. Now,

$$\vec{r}'(t) = \langle 2\alpha t, \beta \cos \beta t, -\beta \sin \beta t \rangle$$

and

$$\vec{r}''(t) = \langle 2\alpha, -\beta^2 \sin \beta t, -\beta^2 \cos \beta t \rangle$$

So that

$$\vec{r}''(t) \cdot \vec{r}'(t) = 4\alpha^2 t$$

It follows that

$$\begin{aligned} W &= \int_0^1 4m\alpha^2 t dt \\ &= 2\alpha^2 m \end{aligned}$$

Example 412 An object acted on by a force $F(x, y) = \langle x^3, y \rangle$ moves along the parabola $y = 3x^2$ from $(0, 0)$ to $(1, 3)$. Calculate the work done by F . Since the curve is not parametrized, we must do so first. For this parabola, we can use $\vec{r}(t) = \langle x(t), y(t) \rangle$ where $x = t$, $y = 3t^2$, $0 \leq t \leq 1$. Now,

$$\begin{aligned} F(\vec{r}(t)) &= F(x(t), y(t)) \\ &= \langle x^3(t), y(t) \rangle \\ &= \langle t^3, 3t^2 \rangle \end{aligned}$$

Also

$$\vec{r}'(t) = \langle 1, 6t \rangle$$

So that

$$\begin{aligned} F(\vec{r}(t)) \cdot \vec{r}'(t) &= t^3 + 18t^3 \\ &= 19t^3 \end{aligned}$$

It follows that

$$\begin{aligned} W &= \int_0^1 19t^3 dt \\ &= \frac{19}{4} \end{aligned}$$

Remark 413 (independence of the parametrization) *This remark applies to all the integrals studied in this section, that is line integrals of scalar functions along plane or space curves as well as line integrals of vector fields. To evaluate these integrals, one must have a parametrization of the curve involved. Since the same curve can be parametrized different ways a natural question is to know if the results depends on the parametrization. It can be proven that it does not. Given a smooth curve and any parametrization for it, a line integral along this curve will have a unique answer, which does not depends on the parametrization chosen.*

5.2.5 Assignment

- Evaluate $\int_C y ds$ where C is the curve given by the position vector $\vec{r}(t) = \langle t^2, t \rangle$, $0 \leq t \leq 2$.
- Evaluate $\int_C xy^4 ds$ where C is the right half of the circle $x^2 + y^2 = 16$.
- Evaluate $\int_C xy dx + (x - y) dy$ where C consists of line segments from $(0, 0)$ to $(2, 0)$ and from $(2, 0)$ to $(3, 2)$.
- Evaluate $\int_C (xy + y + z) ds$ along the curve $\vec{r}(t) = \langle 2t, t, 2 - 2t \rangle$, $0 \leq t \leq 1$.
- Evaluate $\int_C xe^{yz} ds$ where C is the line segment from $(0, 0, 0)$ to $(1, 2, 3)$.
- Evaluate $\int_C x^2 y \sqrt{z} dz$ where C is the curve given by the position vector $\vec{r}(t) = \langle t^3, t, t^2 \rangle$, $0 \leq t \leq 1$.
- Evaluate $\int_{C_1 \cup C_2} (x + \sqrt{y} - z^2) ds$ where C_1 is given by $\vec{r}_1(t) = \langle t, t^2, 0 \rangle$, $0 \leq t \leq 1$ and C_2 is given by $\vec{r}_2(t) = \langle 1, 1, t \rangle$, $0 \leq t \leq 1$.
- Find the mass of the wire that lies along the curve $\vec{r}(t) = \langle 0, t^2 - 1, 2t \rangle$, $0 \leq t \leq 1$ if the density of the wire is given by $f(x, y, z) = \frac{3t}{2}$.
- Find the work done by $F(x, y, z) = \langle 3y, 2x, 4z \rangle$ over each path below:
 - C_1 given by $\vec{r}(t) = \langle t, t, t \rangle$, $0 \leq t \leq 1$.
 - C_2 given by $\vec{r}(t) = \langle t, t^2, t^4 \rangle$, $0 \leq t \leq 1$.
- Find the work done by $F(x, y, z) = \langle 3x^2 - 3x, 3z, 1 \rangle$ over each path below:
 - C_1 given by $\vec{r}(t) = \langle t, t, t \rangle$, $0 \leq t \leq 1$.
 - C_2 given by $\vec{r}(t) = \langle t, t^2, t^4 \rangle$, $0 \leq t \leq 1$.
- Find the work done by $F(x, y, z) = \langle 2y, 3x, x + y \rangle$ over the curve given by $\vec{r}(t) = \left\langle \cos t, \sin t, \frac{t}{6} \right\rangle$, $0 \leq t \leq 2\pi$