The Binomial Series

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Abstract

This hand reviews the binomial theorem and presents the binomial series.

1 The Binomial Series

1.1 The Binomial Theorem

This theorem deals with expanding expressions of the form $(a+b)^k$ where k is a positive integer. In the case $k=2$, the result is a known identity

$$(a+b)^2 = a^2 + 2ab + b^2$$

It is also easy to derive an identity for $k=3$.

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

There is also a formula for k in general. That formula is known as the **Binomial Theorem**. Before we state it, let us explain it a little bit. $(a+b)^k$ will be a sum of terms. Each term will contain a coefficient as well as powers of a and b. More precisely, we will have

$$\begin{aligned}(a+b)^k &= \\ & a^k + ka^{k-1}b + \frac{k(k-1)}{2!}a^{k-2}b^2 + \frac{k(k-1)(k-2)}{3!}a^{k-3}b^3 \\ & +...+\frac{k(k-1)(k-2)...(k-n+1)}{n!}a^{k-n}b^k \\ & +...+kab^{k-1}+b^k\end{aligned}$$

We see from the formula that the powers of a and b are of the form a^ib^j where i decreases from k to 0 and j increases from 0 to k. The coefficients $\frac{k(k-1)(k-2)...(k-n+1)}{n!}$ which appear in this expansion are called **binomial coefficients**. We use a special notation for them.

Definition 1 (Binomial coefficients) *The binomial coefficients, denoted* $\binom{k}{n}$*, are defined by:*

$$\begin{aligned} \binom{k}{n} &= \frac{k(k-1)(k-2)\ldots(k-n+1)}{n!} \text{ if } n \geq 1 \\ \binom{k}{0} &= 1 \end{aligned}$$

Remark 2 *The numerator of the fraction in the definition has exactly n terms. This is helpful when figuring out the coefficients.*

This notation allows us to write:

Theorem 3 (Binomial Theorem) *Suppose that* k *is a positive integer. Then*

$$(a+b)^n = \sum_{n=0}^{k} \binom{k}{n} a^{k-n} b^n$$

Example 4 *Find* $\binom{4}{n}$ *for* $n = 0, 1, 2, 3, 4$.

- $\binom{4}{0} = 1$
- $\binom{4}{1} = 4$
- $\binom{4}{2} = \frac{4 \times 3}{2!} = 6$
- $\binom{4}{3} = \frac{4 \times 3 \times 2}{3!} = 4$
- $\binom{4}{4} = \frac{4 \times 3 \times 2 \times 1}{4!} = 1$

Example 5 *Expand* $(a+b)^4$.
From the binomial theorem, we have

$$\begin{aligned} (a+b)^4 &= \sum_{n=0}^{4} \binom{4}{n} a^{4-n} b^n \\ & \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4 \\ &= a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4 \end{aligned}$$

For what follows, we will be interested in expanding $(1+x)^n$. In the case n is a positive integer, the binomial theorem gives us

$$(1+x)^n = \sum_{n=0}^{k} \binom{k}{n} x^n$$

1.2 The Binomial Series

The binomial series extends the binomial theorem for cases when k is not an integer. For example, how would we expand $(1+x)^{\frac{1}{2}}$? In other words, given $f(x) = (1+x)^k$, for any k, what is a Maclaurin series for f? We derive it like any other Maclaurin series. So, we have

$$(1+x)^k = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

So, we need to find $f^{(n)}(0)$.

$$\begin{array}{rr}
f(x) = (1+x)^k & f(0) = 1 \\
f'(x) = k(1+x)^{k-1} & f'(0) = k \\
f''(x) = k(k-1)(1+x)^{k-2} & f''(0) = k(k-1) \\
f'''(x) = k(k-1)(k-2)(1+x)^{k-3} & f'''(0) = k(k-1)(k-2) \\
\vdots & \vdots \\
f^{(n)}(x) = k(k-1)\ldots(k-n+1)(1+x)^{k-n} & f^{(n)}(0) = k(k-1)\ldots(k-n+1)
\end{array}$$

Therefore, the Maclaurin series for $(1+x)^k$ is

$$\begin{aligned}
(1+x)^k &= \sum_{n=0}^{\infty} k(k-1)\ldots(k-n+1)\frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \binom{k}{n} x^n
\end{aligned}$$

This series is known as the binomial series. To study its convergence, we use the ration test.

$$\begin{aligned}
\frac{|a_{n+1}|}{|a_n|} &= \\
& \frac{|k(k-1)\ldots(k-n)|\,|x|^{n+1}}{(n+1)!}\frac{n!}{|k(k-1)\ldots(k-n+1)|\,|x|^n} \\
&= \frac{|k-n|}{n+1}|x|
\end{aligned}$$

So, since

$$\lim_{n\to\infty}\frac{|k-n|}{n+1} = 1$$

we have

$$\lim_{n\to\infty}\frac{|k-n|}{n+1}|x| = |x|$$

By the ratio test, this series converges if $|x| < 1$. Convergence at the endpoints depends on the values of k and needs to be checked every time.

Definition 6 (Binomial Series) *If $|x| < 1$ and k is any real number, then*

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where the coefficients $\binom{k}{n}$ are the binomial coefficients.

Remark 7 *This formula is very similar to the formula in the binomial theorem. The only difference is that in the binomial theorem, we have a finite sum. In this case, we have an infinite sum.*

Remark 8 *In the case k is a positive integer, this formula is the same as the formula of the binomial theorem. In this case, $\binom{k}{n} = 0$ whenever $k > n$ (why?).*

Example 9 *Expand $\frac{1}{1+x}$ as a power series.*

We have already done this using substitution and the power series of $\frac{1}{1-x}$. We found

$$\begin{aligned} \frac{1}{1+x} &= 1 - x + x^2 - x^3 + ... \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

We just derive the series to illustrate our new technique. From the formula above, we will have

$$\begin{aligned} \frac{1}{1+x} &= (1+x)^{-1} \\ &= \sum_{n=0}^{\infty} \binom{-1}{n} x^n \end{aligned}$$

We compute

$$\binom{-1}{n} = \frac{(-1)(-1-1)(-1-2)...(-1-n+1)}{n!}$$

Remember that the numerator has n factors. So, we get

$$\begin{aligned} \binom{-1}{n} &= \frac{(-1)^n (1)(2)(3)...(n)}{n!} \\ &= (-1)^n \end{aligned}$$

Therefore,

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

Which is the same formula as we had found using a different method.

Example 10 *Expand* $\dfrac{1}{\sqrt{1+x}}$ *as a power series.*
First, we notice that

$$\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$$

So, we use the formula above in the case $k = -\dfrac{1}{2}$. *We obtain:*

$$\frac{1}{\sqrt{1+x}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n$$

where

$$\binom{-\frac{1}{2}}{0} = 1$$

and if $n \geq 1$,

$$\begin{aligned} \binom{-\frac{1}{2}}{n} &= \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)...\left(-\frac{1}{2}-n+1\right)}{n!} \\ &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)...\left(-\frac{2n-1}{2}\right)}{n!} \end{aligned}$$

The numerator has n *factors. We get*

$$\binom{-\frac{1}{2}}{n} = (-1)^n \frac{(1)(3)(5)...(2n-1)}{2^n n!}$$

Therefore,

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{(1)(3)}{2^2 2!}x^2 - \frac{(1)(3)(5)}{2^3 3!}x^3....$$

1.3 Problems

Do # 1, 3, 7, 9 on page 625