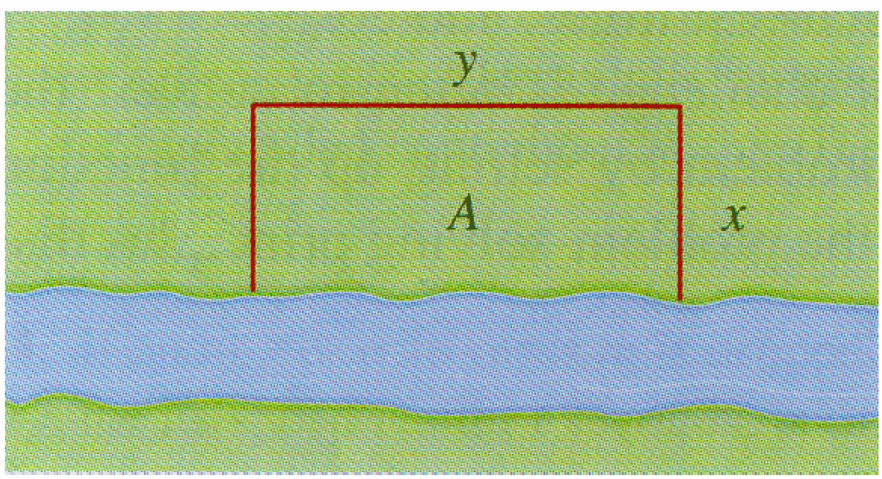
3.8 Optimization problems (4.6)

In these problems, using the methods of calculus, the goal is usually to find the maximum or minimum value of a certain quantity. We will use the tools to find extreme values developed in the previous sections. Since these tools apply to functions of one variable, the first step in these problems will be to express the quantity to maximize or minimize as a function of onbe other variable. We illustrate how this is done with an example. Explanations as well as all the steps to follow will be provided to show the students how such a problem should be handled. Make sure you do not skip any of the steps outlined below when you work on such a problem.

Example 217 *A farmer wants to fence in a rectangular region. A river runs along one of the sides of the region, so the farmer only needs to put the fence on three of the sides. If the farmer has 200 yards of fence to use, find the largest possible area which can be enclosed by the fence. What are the dimensions of the region having this area?*

3.8.1 Understand the problem. Draw a picture if relevant, label the various quantities involved.



Let x and y denote the length of the sides as shown on the picture.
Let A be the area of the region.

3.8.2 Identify the quantity to maximize or minimize

We want to find the maximum value A can have.

3.8.3 Express the quantity to maximize or minimize as a function of one other quantity

Since A is the area of the field, $A = xy$

So, we need to eliminate either x or y in the above relation. On one hand, we know that the farmer has 200 yards of fence to use. On the other hand, looking at the picture, we see that the length of the fence will be $x + y + x$. So, $2x + y = 200$ which is the same as $y = 200 - 2x$

Therefore, $A = x(200 - 2x) = 200x - 2x^2$

So, our problem amounts to finding the maximum value $A = 200x - x^2$ can have.

3.8.4 Use calculus to find the maximum or the minimum

To find the maximum of A, we find its critical numbers, and we test them

$A^{'} = 200 - 4x$.

$A^{'}$ is always defined since it is a polynomial.

$A^{'} = 0$ if $200 - 4x = 0$ that is if $x = 50$.

So, 50 is a critical number. Now, we need to see if it is a maximum. We use the second derivative test. $A^{''} = -4 < 0$. So, in particular, $A^{''}\Big|_{x=50} = -4 < 0$. So, by the second derivative test, A has a maximum when $x = 50$

3.8.5 Check the answer makes sense, put it in the required form

The first part of the question asked to find the largest area that could be enclosed. In other words, we had to find the maximum value of A.

We just found that A was maximum when $x = 50$. Since $A = 200x - x^2$, when $x = 50$, $A = 5000$. so, the largest area which can be enclosed is 5000 square yards.

The second part of the question asked the dimension of the region having this largest area. The largest area is attained when $x = 50$. Since $y = 200 - 2x$, $y = 100$. So, the dimensions of the region are 50×100, the side which measures 100 yards is the side parallel to the river.

3.8.6 Examples

Example 218 *Find two numbers whose sum is 100 and product is the largest.*
Let x and y denote the two numbers, let P denote their product. We want to find the largest value P can have, given that $x + y = 100$.
The various quantities involved are P, x and y. We want to maximize P. First, we need to express P as a function of one of the other quantities.
We know that $P = xy$. We need to eliminate either x or y. Since $x + y = 100$, $y = 100 - x$. Thus, $P = x(100 - x) = 100x - x^2$.
We want to find the maximum of $P(x) = 100x - x^2$. Since a maximum can only happen at a critical number, we find the critical numbers of P. $P'(x) = 100 - 2x$.

Thus P has one critical number, $x = 50$. Since $P''(x) = -2 < 0$, by the second derivative test, we see that this critical number corresponds to a maximum. Thus, P has a maximum when $x = 50$.
Finally, we put the answer in the required form. We had to find the two numbers whose product was maximum. We found that the product was maximum when $x = 50$. Since $x + y = 100$, it follows that $y = 50$. The two numbers are 50 and 50.

Example 219 *Find the largest area a rectangular region can have if its perimeter is 200.*
Let x and y be the dimensions of the rectangular region. Let A be its area. We want to find the largest value A can have, given that the perimeter of the region is 200. The quantities involved are A, x, y; we want to maximize A. The first step is to express A as a function of another quantity. We know that $A = xy$. We need to eliminate y. We can use the fact that $2(x + y) = 200$ that is $x + y = 100$. Therefore, $A = x(100 - x) = 100x - x^2$
So, the problem is to maximize $A = 100x - x^2$. This is the same problem as the previous one. We found that A was maximum when $x = y = 100$. We see that the rectangular region having the largest area is a square.

Example 220 *Find the area of the largest rectangle that can be inscribed in a semi circle of radius r.*
For simplicity, we consider the upper half of the circle of radius r, centered at the origin. Its equation is $x^2 + y^2 = r^2$ with $y \geq 0$. Thus $y = \sqrt{r^2 - x^2}$. We label (x, y) the upper right corner of the rectangle. Thus, the rectangle has height y and width $2x$. Let A denote the area of this rectangle. Then, $A = 2xy$. We want to maximize A. First, we express A as a function of one other quantity.

$$\begin{aligned} A(x) &= 2xy \\ &= 2x\sqrt{r^2 - x^2} \end{aligned}$$

Now, we use calculus to find the maximum A can have.

$$\begin{aligned} A' &= 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} \\ &= \frac{2\left(r^2 - 2x^2\right)}{\sqrt{r^2 - x^2}} \end{aligned}$$

The critical numbers are $x = \frac{r}{\sqrt{2}}$. Using the first derivative test, we see that this critical number corresponds to a maximum.
Thus, the largest area happens when $x = \frac{r}{\sqrt{2}}$. The area is $A\left(\frac{r}{\sqrt{2}}\right)$

$$\begin{aligned} A\left(\frac{r}{\sqrt{2}}\right) &= 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} \\ &= r^2 \end{aligned}$$

3.8.7 Things to know

- Be able to use Calculus to solve optimization problems
- Be able to do # 3, 5, 7, 9, 11, 13, 15, 33 on pages 311-315.
- Be able to do problems like the problems below:

1. A ball is thrown straight upward from the ground; t seconds later, its height in feet is $h(t) = 48t - 16t^2$. What is the maximum height reached by the ball? (**Answer**: maximum altitude is 35.56 feet, when $t = 4/3$)
2. A manufacturer estimates that the profit from production of x units of a certain item is $P(x) = 40x - 0.02x^2 - 600$. How many units should be produced for maximum profit? (**Answer**: 1000 units should be produced for a maximum profit of 19400)
3. A box with square base, rectangular sides and no top is to be made from 27 ft^2 of cardboard. What is the largest volume of such a box? (**Answer**: Maximum volume is 13.5 and the dimension of the box are $3 \times 3 \times 1.5$)
4. A rectangle with base on the x-axis has its upper vertices on the curve $y = 3 - x^2$.
   (a) Find the maximum area of such a rectangle. (**Answer**: Maximum area is 4, when $x = 1$)
   (b) Does a rectangle of minimum area exist? What is the range of possible areas? (**Answer**: No)
5. Repeat problem 4 with "perimeter" in place of "area". (**Answer**: Maximum perimeter is 8, when $x = 1$)
6. A rancher intends to fence off a rectangular region along a river (which serves as a natural boundary requiring no fence). If the enclosed area is to be 1800 square yards, what is the least amount of fence needed? (**Answer**: 120 yards of fence is needed for a region 30×60, 60 being parallel to the river).