Trigonometric Functions and Triangles

Dr. Philippe B. Laval
Kennesaw STate University

August 27, 2010

Abstract

This handout defines the trigonometric function of angles and discusses the relationship between trigonometric functions and triangles. This covers the material presented in sections 5.1 - 5.4 in your book.

1 Review: Angles

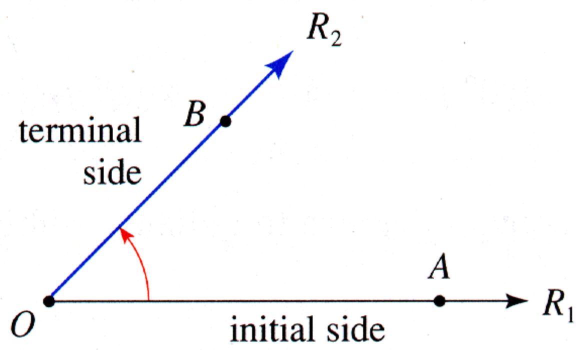
This section is about the following concepts:

- Angles
- Radian and degree measure of an angle
- Coterminal and reference angles
- Length of an arc
- Area of a circular sector

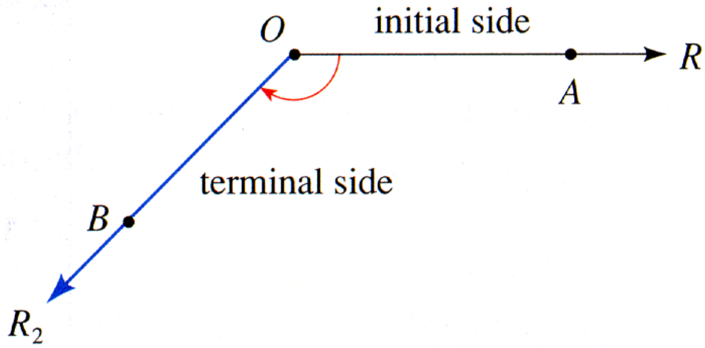
If you already know these concepts, you can skip to the next section on page 10. If you are not entirely sure, try the practice problems on page 9.

1.1 Angle Measure

An angle is determined by rotating a ray about its end point, called the vertex. The ray in the starting position is called the initial side. The ray is the final position is called the terminal side. The measure of the angle is the amount of rotation. If the rotation is counterclockwise, the measure of the angle will be positive; if it is clockwise, the measure of the angle will be negative. We often use greek letters such as α (alpha), β (beta) or θ (theta) to name angles. The pictures below illustrates these definitions.



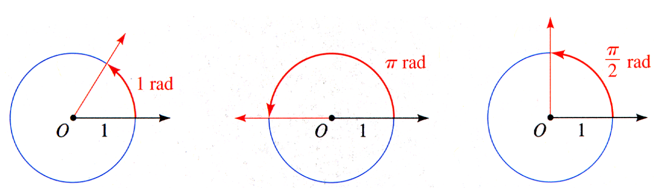
Positive angle



Negative angle

Definition 1 (Radian Measure) *If a circle of radius 1 (unit circle) is drawn with the vertex of an angle at its center, then the measure of this angle in radians (rad) is the length of the arc that subtends the angle.*

Since the circumference of a unit circle is 2π, the measure of the angle of a complete revolution is 2π rad.



Definition 2 (Degree Measure) *The definition is similar, the difference is that in degrees, a complete revolution is* $360°$.

To convert between degrees and radians, we use the fact $180^\circ = \pi$ rad. From this, we see that

$$\begin{aligned} 1 \text{ rad} &= \left(\frac{180}{\pi}\right)^\circ \\ 1^\circ &= \frac{\pi}{180} \text{ rad} \end{aligned}$$

Therefore, we have the following proposition.

Proposition 3 *How to convert an angle measure from degrees to radians or vice-versa.*

1. *To convert the radian measure of an angle into degrees, multiply it by* $\frac{180}{\pi}$.
2. *To convert the degree measure of an angle into radians, multiply it by* $\frac{\pi}{180}$.

Example 4 *Convert 40 degrees to radians.*
Since $1^\circ = \frac{\pi}{180}$ *rad, it follows that* $40^\circ = 40\frac{\pi}{180}$ *rad that is* $40^\circ = \frac{2\pi}{9}$ *rad.*

Example 5 *Convert* $\frac{\pi}{3}$ *rad to degrees.*
Since 1 *rad* $= \left(\frac{180}{\pi}\right)^\circ$, *it follows that* $\frac{\pi}{3} = \frac{\pi}{3}\frac{180}{\pi} = 60^\circ$.

1.1.1 Decimal degrees versus minutes and seconds.

Fractional degrees can be expressed two ways. We can use decimal degrees. For example, 30.6° means 30° plus $\frac{6}{10}$ of a degree. One can also use minutes (denoted $'$) and seconds (denoted $''$). They are defined as follows:

$$\begin{aligned} 1^\circ &= 60' \\ 1' &= 60'' \end{aligned}$$

Historically, degrees, minutes and seconds (DMS) were used. With scientific calculators, decimal degrees are being used more and more. We illustrate switching from one notation to the other with examples.

Example 6 *Write* $30^\circ 22' 48''$ *in decimal degrees notation.*

- *First, we convert the seconds into minutes.*

  $$1' = 60'' \iff 1'' = \frac{1}{60}'$$

  So,

  $$\begin{aligned} 48'' &= \frac{48'}{60} \\ &= .8' \end{aligned}$$

  Therefore, $30^\circ 22' 45'' = 30^\circ 22.8'$

- *Next, we convert $22.8'$ into degrees using a similar technique.*

$$1^\circ = 60' \iff 1' = \frac{1}{60}^\circ$$

So

$$\begin{aligned} 22.8' &= \frac{22.8}{60}^\circ \\ &= .38^\circ \end{aligned}$$

Therefore, $30^\circ 22' 45'' = 30^\circ 22.8' = 30.38^\circ$

Remark 7 *To convert DMS to decimal degrees, we divide the number of seconds by 60 and add the result to the number of minutes. This gives us the decimal minutes. We then divide the decimal minutes by 60 and add the result to the number of degrees. This gives us the angle measure using decimal degree notation.*

Example 8 *Convert 30.38° to DMS notation.*
We use the fact that $1^\circ = 60'$ therefore, $.38^\circ = .38 * 60' = 22.8'$ (that is $22'$ and $.8'$). Now, we convert the decimal minutes to seconds using a similar technique. $1' = 60''$ therefore $.8' = .8 * 60'' = 48''$. It follows that $30.38^\circ = 30^\circ 22' 48''$.

Remark 9 *Scientific calculators can also do this automatically. See the manual for your calculator to learn how to do it.*

1.2 Angles, Points and the Unit Circle

Definition 10 (Unit Circle) *The **Unit Circle** is the circle of radius 1, centered at the origin (the point with coordinates $(0, 0)$).*

The equation of the unit circle is

$$x^2 + y^2 = 1$$

This means that a point with coordinates (x, y) is on the unit circle if and only if its coordinates satisfy $x^2 + y^2 = 1$.

Example 11 *Is the point with coordinates $(1, 2)$ on the unit circle?*
we see that $1^2 + 2^2 = 1 + 4 = 5 \neq 1$. Therefore, the point with coordinates $(1, 2)$ is not on the unit circle.

Example 12 *Is the point with coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle?*

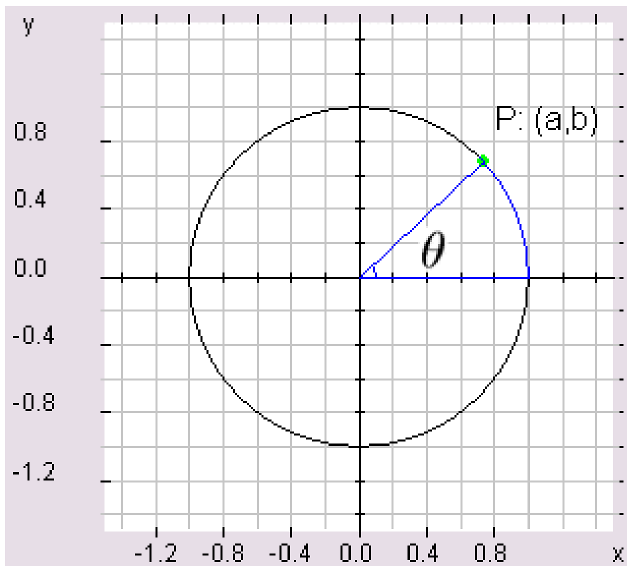
We see that

$$\begin{aligned} \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 &= \frac{2}{4} + \frac{2}{4} \\ &= \frac{4}{4} \\ &= 1 \end{aligned}$$

Therefore, the point with coordinates $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ *is on the unit circle.*

Definition 13 (Angles in standard position) *An angle is in standard position if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis.*

The picture below shows an angle in standard position.



Given an angle θ in standard position, its terminal side intersects the unit circle at a point P. We will denote by (a, b) the coordinates of P. One of the goals of this chapter is to find the relationship between an angle and the coordinates of the corresponding point on the unit circle. The purpose of this section is to show that it is enough to know this relationship in the first quadrant. Once we know it in the first quadrant, we know it for the whole circle. We begin with some definitions and remarks.

1.2.1 Coterminal Angles

Definition 14 (Coterminal angles) *Two angles are coterminal if they have the same sides.*

This means that coterminal angles will produce the same point Pon the unit circle.

Given an angle θ, all the angles coterminal to it will be of the form $\theta + 360k$ (in degrees) or $\theta + 2k\pi$ (in radians), where k is any integer. This should be obvious since we are adding or subtracting complete revolutions.

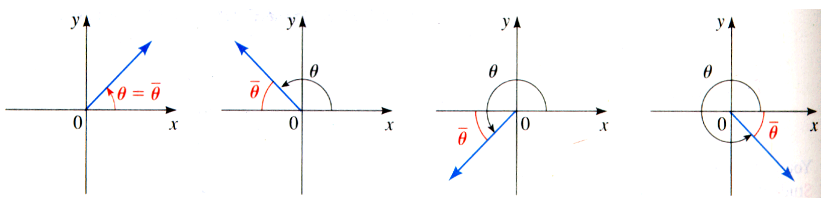
Example 15 *Find an angle coterminal with* $35°$
We can find such an angle by adding or subtracting any multiple of a complete rotation (360°)*. Thus* $35 + 360 = 395$ *is a possible answer. So is* $35 + 2 * 360 = 755$ *or* $35 - 360 = -325$*...*

Example 16 *Find an angle coterminal with* 900° *whose value is between* 0 *and* 360*.*
To do this, we subtract 360 until the angle is within the desired range. We can even figure out which multiple of 360 *has to be subtracted by dividing 900 by 360.* $\frac{900}{360} = 2.5$*. So, the angle we want is* $900 - 2(360) = 180°$*. You will note that it is simply the remainder of the division of* 900 *by* 360*.*

1.2.2 Reference Angles

Definition 17 *The reference angle* $\overline{\theta}$ *of an angle* θ *in standard position is the acute angle (between* 0 *and* $90°$*) the terminal side makes with the x-axis.*

The picture below illustrates this concept.



To find the reference angle associated with an angle θ, follow the procedure below:

1. Find the angle coterminal with θ whose value is between 0 and 360. Call it θ'. (If θ is already between 0 and 360, then $\theta' = \theta$).

2. The value of $\overline{\theta}$ depends on which quadrant θ' is. The table below shows how to find it.

Quadrant for θ'	How to find $\overline{\theta}$ in degrees	How to find $\overline{\theta}$ in radians
I	$\overline{\theta} = \theta'$	$\overline{\theta} = \theta'$
II	$\overline{\theta} = 180 - \theta'$	$\overline{\theta} = \pi - \theta'$
III	$\overline{\theta} = \theta' - 180$	$\overline{\theta} = \theta' - \pi$
IV	$\overline{\theta} = 360 - \theta'$	$\overline{\theta} = 2\pi - \theta'$

Remark 18 *A reference angle is an angle in the first quadrant, that is its value is between* 0 *and* $90°$ *or between* 0 *and* $\frac{\pi}{2}$ *radians.*

Example 19 *Find the reference angle associated with* $295°$
295 *is in the fourth quadrant, thus the reference angle associated with it is* $360 - 295 = 65°$

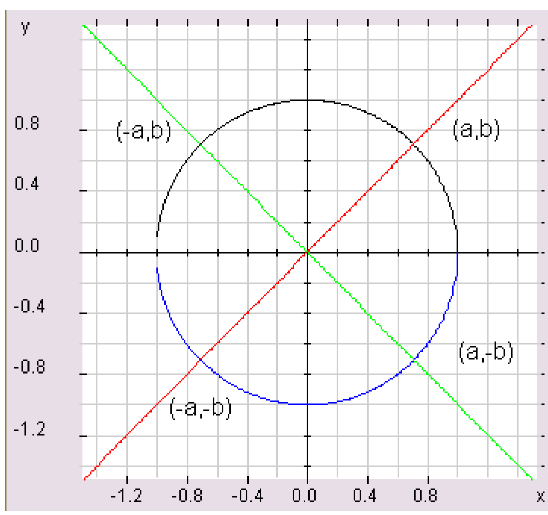
Example 20 *Find the reference angle associated with* $905°$
First, we need to find the angle coterminal to 905, whose value is between 0 *and* 360. *Since* $\frac{905}{360} = 2.5139$, *the angle we want is* $905 - 2(360) = 185$. 185 *is in the third quadrant, thus the reference angle associated with it is* $185 - 180 = 5°$

1.2.3 Applications of Reference Angles

Reference angles allow us to find the coordinates of any point P associated with an angle θ assuming we know how to do this in the first quadrant. The table below explains how to do this. Let θ denote an angle and $\overline{\theta}$ its corresponding reference angle. Let P be the point associated with θ and $\overline{P}$ be the point associated with $\overline{\theta}$.

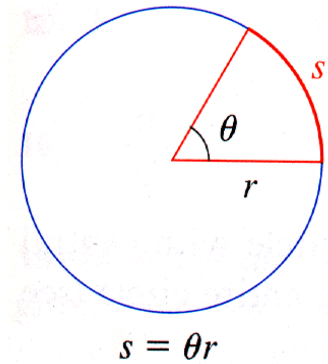
Quadrant of θ	If $\overline{P}$ is	Then P will be
I	(a, b)	(a, b)
II	(a, b)	$(-a, b)$
III	(a, b)	$(-a, -b)$
IV	(a, b)	$(a, -b)$

This is due to the relationship which exists between $P$and $\overline{P}$ depending on which quadrant P is in. The picture below illustrates this relationship.



In conclusion, we see that the coordinates of a point associated with an angle and the coordinates of the point associated with its reference angle are the same, except for their sign. Their sign being determined by the quadrant in which the terminal side of the angle is.

1.3 Length of an arc in a circle of radius r



An angle whose radian measure is θ is subtended by an arc that is the fraction $\dfrac{\theta}{2\pi}$ of the circumference of the circle. Thus, using the notation of the picture above, we have

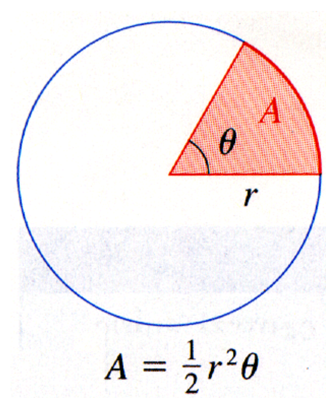
$$\begin{aligned} s &= \frac{\theta}{2\pi} \times \text{ circumference of the circle} \\ &= \frac{\theta}{2\pi} 2\pi r \\ s &= \theta r \end{aligned}$$

If we solve for θ, we get an expression for the measure of an angle, in any circle, not just unit circles.

$$\theta = \frac{s}{r}$$

Remark 21 *This formula is only valid when using radian measure.*

1.4 Area of a sector in a circle of radius r



The area of a circle of radius r is πr^2. The area of the sector A on the picture is the fraction $\dfrac{\theta}{2\pi}$ of the area of the entire circle. Thus,

$$\begin{aligned} A &= \frac{\theta}{2\pi} \times \text{ area of the circle} \\ &= \frac{\theta}{2\pi}\pi r^2 \\ A &= \frac{1}{2}\theta r^2 \end{aligned}$$

1.5 Practice Problems

1. Find the radian measure of the angle with the given degree measure.

   (a) 40
   (b) 330
   (c) -30
   (d) 45
   (e) 36

2. Find the degree measure of the angle with the given radian measure.

   (a) $\dfrac{\pi}{3}$.
   (b) $\dfrac{\pi}{6}$.
   (c) $\dfrac{\pi}{4}$.
   (d) $\dfrac{2\pi}{9}$.
   (e) $\dfrac{7\pi}{3}$.
   (f) 1.5

3. The measure of an angle in standard position is given below. Find the measure of two positive and two negative angles that are coterminal with the given angle.

   (a) $300°$.
   (b) $135°$
   (c) $\dfrac{3\pi}{4}$ rad
   (d) $-50°$

4. The measures of two angles are given. Determine if they are coterminal or not.

(a) $70°$ and $430°$

(b) $\frac{5\pi}{6}$ and $\frac{17\pi}{6}$

(c) $\frac{3\pi}{2}$ and $990°$

5. Find the angle between 0 and 360°coterminal with the given angle

   (a) $733°$

   (b) $-800°$

   (c) $361°$

   (d) $1270°$

6. Find the length of the arc that subtends a central angle of 45° in a circle of radius 10 meters

7. Find the radius of the circle if an arc of length 6 meters on the circle subtends a central angle of $\frac{\pi}{6}$ rad

8. Find the area of a sector with central angle 1 rad in a circle of radius 10 meters.

9. Do # 31, 33, 35, 43, 45 on page 422.

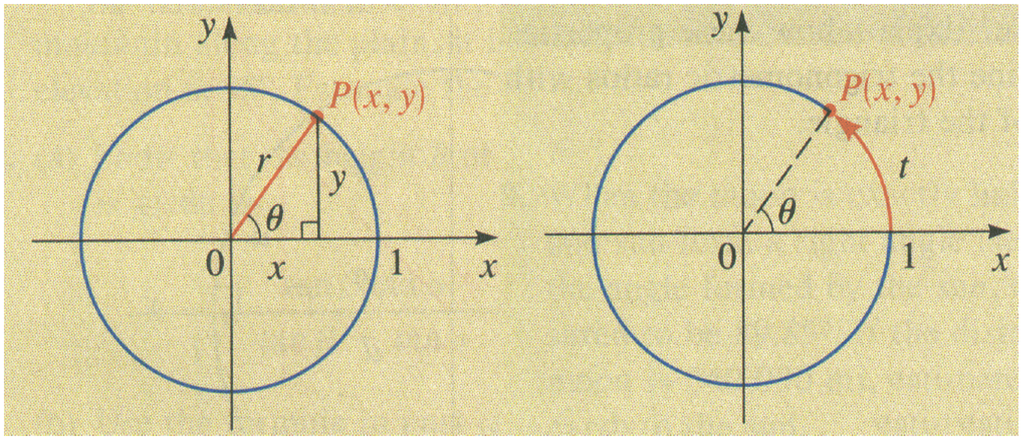
10. Do # 1, 5, 9, 11, 13, 17, 41, 47, 51 on page 447.

11. Do # 1, 2, 3, 5, 9, 11, 21, 23, 33, 35 on pages 461, 462.

2 Trigonometric Functions

2.1 Trigonometric Functions Defined in the Unit Circle

Let θ denote an angle in standard position in the unit circle centered at the origin. Let P be the point of intersection between the circle and the terminal side of θ. The coordinates of P will be called (x, y). Because we are in a unit circle, θ is also the length of the arc between the point with coordinates $(1, 0)$ and P. In other words, to use the terminology of Chapter 5, P is the terminal point determined by θ. Or, in the picture below, $t = \theta$.



We define the trigonometric functions in terms of θ as follows:

$\cos\theta = x$	$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{1}{x}$
$\sin\theta = y$	$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{1}{y}$
$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{x}{y}$

Evaluating the trigonometric functions for any angle is usually done using a calculator. However, students should know the trigonometric functions of the angles given by the table below:

θ in degrees	0	30	45	60	90
θ in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

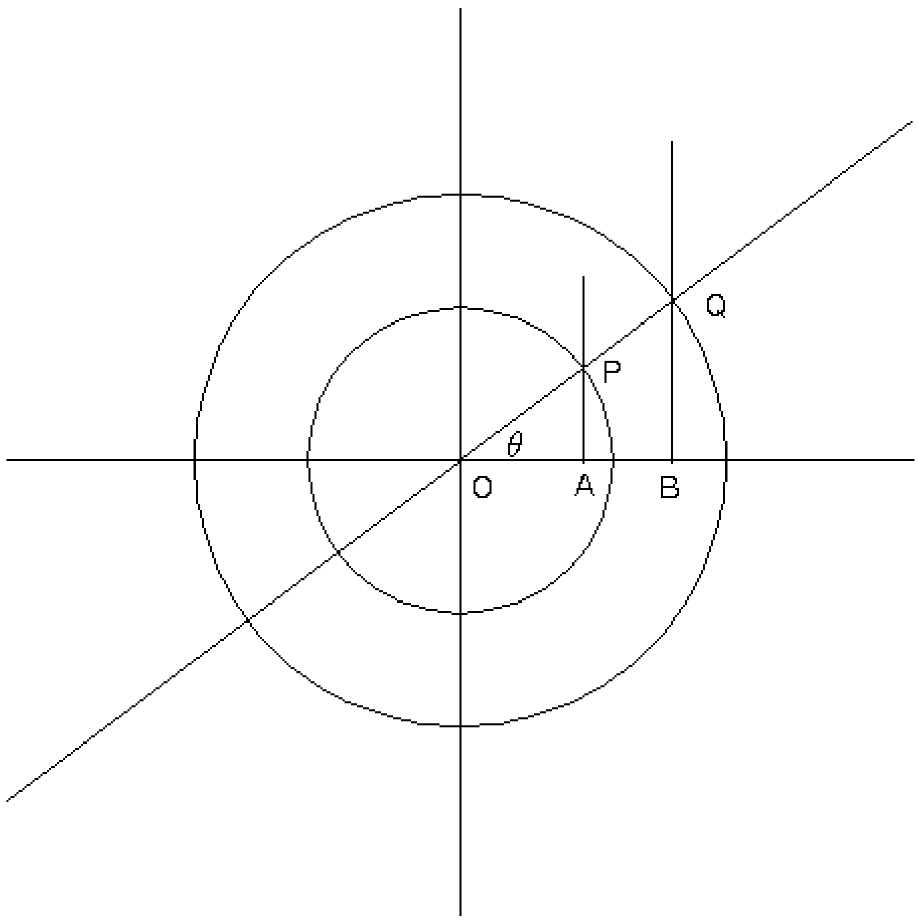
Using reference angles, one can also find the value of the trigonometric functions in the other quadrants.

Example 22 *Find* $\sin 120$ *and* $\cos 120$.
Since $120°$ *is an angle in the second quadrant, its reference angle is* $180 - 120 = 60°$. *The point associated with this angle has coordinates* $(\cos 60, \sin 60) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. *Therefore, the point associated with* $120°$ *has coordinates* $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ *since it is in the second quadrant. Its coordinates are also* $(\cos 120, \sin 120)$. *It follows that* $\cos 120 = -\frac{1}{2}$ *and* $\sin 120 = \frac{\sqrt{3}}{2}$.

It is important to remember that in the above definitions, the numbers x and y are the coordinates of the terminal point on the unit circle. In the next section, we generalize these definitions to any circle.

2.2 Trigonometric Functions Defined in Any Circle

Consider the following picture.



θ is an angle in standard position. Let us assume that the inner circle is a unit circle, the outer circle is a circle of radius r. Therefore, $OP = 1$, $OQ = r$. Let us assume that the coordinates of P are (a, b). This means that $OA = a$, $AP = b$. It also means, from the previous section that $\sin\theta = b$, $\cos\theta = a$.

Let us now assume that the coordinates of Q are (x, y). Then, $OB = x$, $BQ = y$. Also, $x^2 + y^2 = r^2$. Since the triangles OAP and OBQ are similar, we have

$$\frac{OA}{OP} = \frac{OB}{OQ}$$

That is

$$\frac{a}{1} = \frac{x}{r}$$

Since $\cos\theta = a$, it follows that $\cos\theta = \dfrac{x}{r}$. Similarly,

$$\frac{AP}{OP} = \frac{BQ}{OQ}$$

That is

$$\frac{b}{1} = \frac{y}{r}$$

Since $\sin\theta = b$, it follows that $\sin\theta = \dfrac{y}{r}$. Once we have defined $\sin\theta$, and $\cos\theta$, the remaining four trigonometric functions can also be defined.

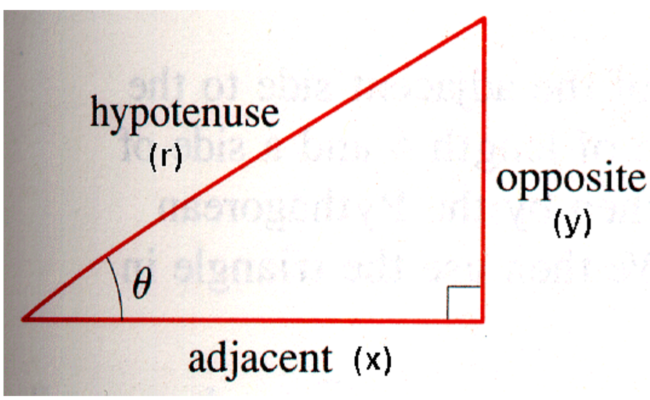
Definition 23 (Trigonometric functions of angles) *Let θ be an angle in standard position, and let $P(x, y)$ be a point on its terminal side. If the distance between P and the origin is r, that is $r = \sqrt{x^2 + y^2}$, then we have*

$\cos\theta = \dfrac{x}{r}$	$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$
$\sin\theta = \dfrac{y}{r}$	$\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$
$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{x}{y}$

Remark 24 *Because of the properties of similar triangles, the quotients $\dfrac{x}{r}$ and $\dfrac{y}{r}$ remain constant. Therefore, the definition of the trigonometric functions does not depend on the point P we select. It only depends on the angle θ.*

2.3 Trigonometric Functions Defined in a Right Triangle

Consider the triangle below:



From the previous section, we have:

$\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$	$\sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent}}$
$\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$	$\csc\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite}}$
$\tan\theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$	$\cot\theta = \frac{x}{y} = \frac{\text{adjacent}}{\text{opposite}}$

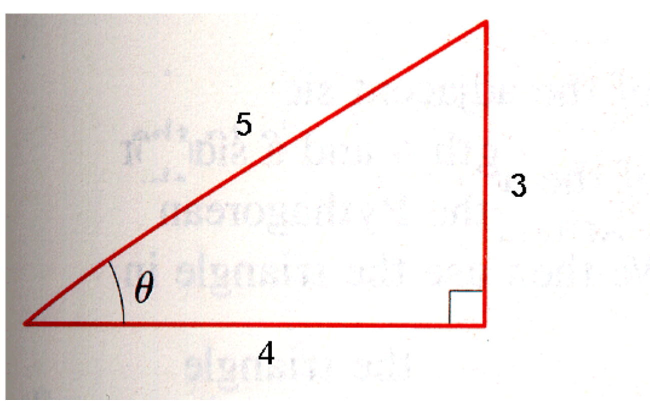
One of the applications of trigonometric functions is related to triangles. A triangle has six parts: 3 angles and 3 sides. **Solving a triangle** means finding all six parts from the given information about the triangle. In this section, we concentrate on right triangles. In the next, we will deal with triangles in general. The case of a right triangle is easier, one of the angles is already known.

2.3.1 Finding Trigonometric Ratios Using Triangles

This type of problem involves finding the value of all the trigonometric functions of an acute angle given one of them. The procedure is as follows:

- Draw a triangle in which the trigonometric function is as given.
- Find the unknown side of the right triangle using the Pythagorean theorem.
- Find the remaining trigonometric ratios.

Example 25 *Find the six trigonometric ratios of the angle θ given that $\tan\theta = \frac{3}{4}$*
Since $\tan\theta = \frac{opposite}{adjacent}$, we draw a triangle in which the side opposite θ is 3, the side adjacent θ is 4. By the Pythagorean theorem, the length of the hypotenuse is 5. Such a triangle is shown below.



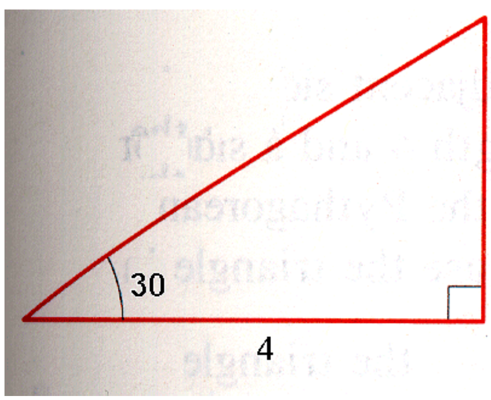
It follows that

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{3}{5}$$
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{4}{5}$$
$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{5}{4}$$
$$\csc\theta = \frac{hypotenuse}{opposite} = \frac{5}{3}$$
$$\cot\theta = \frac{adjacent}{opposite} = \frac{4}{3}$$

2.3.2 Solving Triangles

In order to be able to solve a right triangle, we need to be given two of the parts (we are really given three, since one is already known).

Example 26 *Solve the triangle below*



- *First, we need to determine the angle opposite* $30°$*, call it* β*. Since the sum of the angles in a triangle equals* $180°$*, we have*

$$\begin{aligned} 30 + 90 + \beta &= 180 \\ \beta &= 60° \end{aligned}$$

- *Let* x *denote the length of the side opposite* 30*, and* r *denote the length of the hypotenuse We can find these quantities by using the trigonometric ratios. For example, we have*

$$\cos 30 = \frac{4}{r}$$

Therefore,

$$r = \frac{4}{\cos 30}$$
$$= \frac{4}{\frac{\sqrt{3}}{2}}$$
$$= \frac{8}{\sqrt{3}}$$
$$r = \frac{8\sqrt{3}}{3}$$

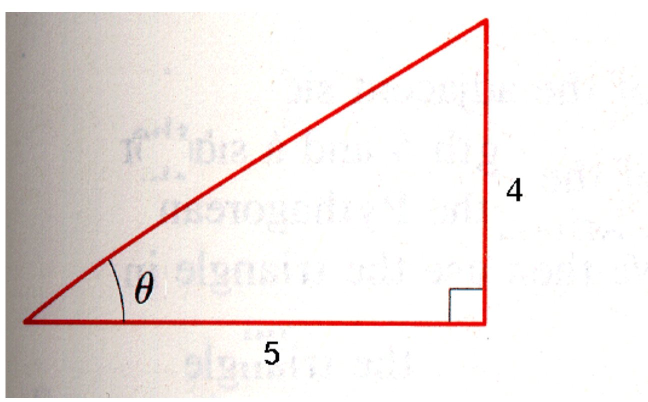
We can now find the remaining side by either using another trigonometric ratio, or the Pythagorean theorem. For example,

$$\sin 30 = \frac{3x}{8\sqrt{3}}$$

Therefore,

$$x = \frac{8\sqrt{3}}{3}\sin 30$$
$$x = \frac{4\sqrt{3}}{3}$$

Example 27 *Solve the triangle below:*



- *Let r be the length of the hypotenuse. Using the Pythagorean theorem, we have*

$$5^2 + 4^2 = r^2$$
$$r^2 = 41$$

Therefore

$$r = \sqrt{41}$$

- *We need to find the two remaining angles. For example, we can write*

$$\tan\theta = \frac{4}{5}$$

The problem is to find θ such that $\tan\theta = \frac{4}{5}$. We will learn how to do this later. For now, we will use our calculator. To find θ such that $\tan\theta$ has a given value, we use the key $\tan^{-1}$. More precisely, $\tan\theta = \frac{4}{5}$, then $\theta = \tan^{-1}\frac{4}{5}$. Simply ask your calculator to compute $\tan^{-1}\frac{4}{5}$. Be careful, the answer is supposed to be an angle. If your calculator is set to work in radians, it will give you an answer in radians. If it is set to work in degrees, it will give an answer in degrees. Since it is easier to visualize angles in degrees, we set our calculator in degrees and obtain

$$\theta = 38.66°$$

- *Let β be the angle opposite θ. Since*

$$\beta + 38.66 + 90 = 180$$

it follows that

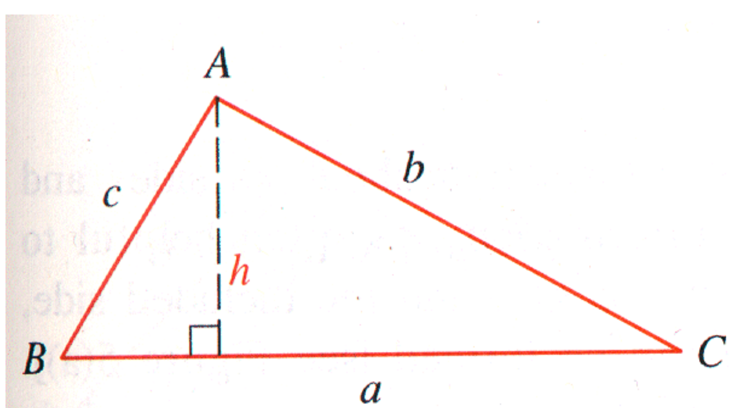
$$\beta = 51.34°$$

Remark 28 *We use a similar method if one of the other trigonometric function is known, to find the angle θ. This is summarized in the table below:*

If we know	*Then*
$\sin\theta = r$	$\theta = \sin^{-1} r$
$\cos\theta = r$	$\theta = \cos^{-1} r$
$\tan\theta = r$	$\theta = \tan^{-1} r$

2.4 Practice Problems

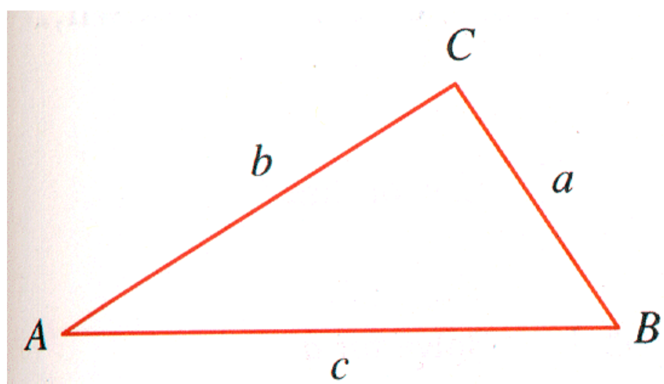
- Looking at the triangle below, express its area in terms of a, b, and C. This is a formula you should remember.



- Related problems assigned in your book (section 5.1 - 5.4):
  - (From 5.1 in the book) Do # 1, 3, 7, 9, 11, 13, 15, 17, 19, 23, 31, 33, 35, 43, 45 on pages 421 - 422.
  - (From 5.2 in the book) Do # 1, 2, 3, 5, 8, 18, 20, 21 on pages 430, 431.
  - (From 5.3 in the book) Do # 1, 5, 9, 11, 13, 17, 25, 27, 28, 29, 31, 33, 34, 41, 47, 51 on page 447.
  - (From 5.4 in the book) Do # 1, 2, 3, 5, 9, 11, 21, 23, 33, 35 on pages 461 - 462.

3 The Laws of Sines and Cosines (skip)

We give these two important laws and some of their applications without proof. These two laws are useful in solving oblique triangles, that is triangles with no right angles. We adopt the following convention. We label the angles of the triangles A, B, C, and the lengths of the corresponding opposite sides a, b, c as shown on the picture below.



To solve this triangle, we need to be given some information. More precisely, a triangle is determined by three of its six parts, as long as at least one is a side. So, we have the following possibilities:

Case 1 **AAA**. This means that we are given three angles. This is the only case when there are infinitely many possibilities. Can you explain why?

Case 2 **SAA**. We are given one side and two angles. This determines the triangle uniquely. Try some cases.

Case 3 **SSA.** We are given two sides and the angle opposite one of the sides. This is the only case when the construction may not always be possible, or there may be a unique possibility, or there may be two. See figure 6, on page 437 in your book.

Case 4 **SAS**. We are given two sides and the included angle. This determines the triangle uniquely. Try some cases.

Case 5 **SSS**. We are given three sides. This determines the triangle uniquely. Try some cases.

Depending on which information is available, we use either the law of sines or the law of cosines to solve the triangle. We now state these two laws, and illustrate them with a few examples.

Theorem 29 (The law of sines) *In a triangle as above, we have*

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

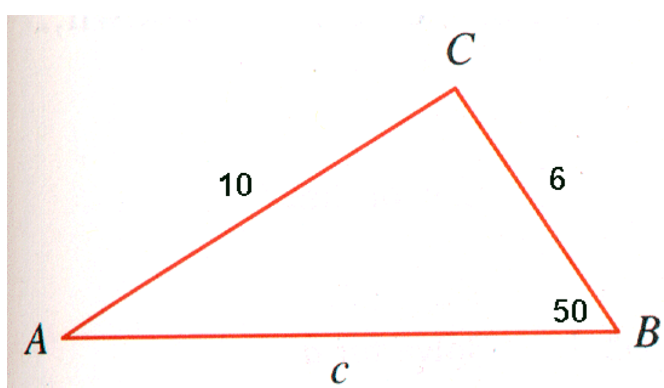
Theorem 30 (The law of cosines) *In a triangle as above, we have*

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

Remark 31 *Each formula in the law of cosines allows you to find the length of a side knowing the angle opposite that side, and the length of the other two sides. It can also be used to find the angles (or the cosine of the angles) of a triangle, knowing its sides.*

Remark 32 *The law of sines is really three formulas. For each formula to be usable, we need to either know two sides and one angle opposite one of the sides. In this case, the formula will give us the other angle. Or, we need to know two angles and a side opposite one of the known angles. In this case, the formula will give us the other side.*

Example 33 *Solve the triangle below*



Since we know that $B = 50$, $b = 10$ and $a = 6$, to solve this triangle, we must find A, C, and c. We need to decide which quantity to find first, and how to do it. If we wanted to find c, since we already know a and b, we might think about the law of cosines. However, we would need to know C, which we don't . We could not use the law of sines either, we would need to know two angles. So, we cannot begin by c. We can either find A or B. Since we know two sides and one angle, we see that the law of sines is appropriate.

- *Finding A. We use*

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$
$$\frac{\sin A}{6} = \frac{\sin 50}{10}$$

$$\begin{aligned} \sin A &= \frac{6 \sin 50}{10} \\ &\approx .4596 \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \sin^{-1} .4596 \\ &\approx 27.36 \end{aligned}$$

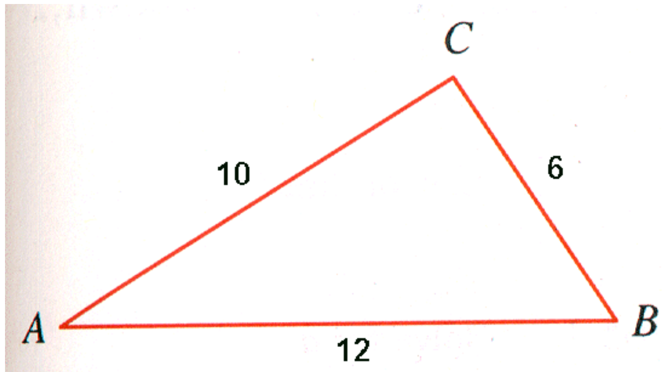
- *Finding C. Now that we know A and B, finding C is easy since the three angles must add up to 180. We solve*

$$\begin{aligned} A + B + C &= 180 \\ C &= 180 - A - B \\ &= 180 - 50 - 27.36 \\ &= 102.64 \end{aligned}$$

- *Finding c. This is the only unknown left, we can find it different ways. We could use the law of cosines. Since $c^2 = a^2 + b^2 - 2ab\cos C$ and c is the only unknown. We could also use one of the laws of sines. In $\frac{\sin B}{b} = \frac{\sin C}{c}$, c is the only unknown, so we can find it.*

$$\begin{aligned} \frac{\sin B}{b} &= \frac{\sin C}{c} \\ c &= \frac{b \sin C}{\sin B} \\ &= \frac{10 \sin 102.64}{\sin 50} \\ &\approx 12.73 \end{aligned}$$

Example 34 *Solve the triangle below*



Here, we know the three sides, we need to find the three angles. We cannot use the law of sines, for that we need to know at least one angle. We will use the law of cosines.

- *Finding A. We write the law of cosines which involves A.*

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{a^2 - b^2 - c^2}{-2bc}$$

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{100 + 144 - 36}{240} \\ &= \frac{13}{15} \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \cos^{-1}\frac{13}{15} \\ &\approx 29.93 \end{aligned}$$

- *Finding B. Now, we can either use the law of sines or the law of cosines. (Give arguments for or against each). We will stick to the law of cosines.*

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{36 + 144 - 100}{144} \\ &= \frac{5}{9} \end{aligned}$$

Therefore,

$$\begin{aligned} B &= \cos^{-1}\frac{5}{9} \\ &\approx 56.25 \end{aligned}$$

- *Finding C.*

$$\begin{aligned} A+B+C &= 180 \\ C &= 180 - A - B \\ C &\approx 93.82 \end{aligned}$$