

Integration by Parts: Another Example of Voodoo Mathematics

Preamble

Of course integration by parts isn't voodoo mathematics; but the way many instructors and almost all text books present it, it is precisely that.

This note is an appeal for integrity in the way we present ourselves and our discipline to our students. It is an appeal on behalf of the following principle:

Mathematics becomes harder, not easier, when we talk nonsense.

In this note I assert that those who present integration by parts in the way that it appears in most texts may not have thought through the topic sufficiently and I appeal to those who teach calculus to take a closer look.

The Integration by Parts Theorem

If f and g are differentiable functions on an interval $[a, b]$ and if the functions f' and g' are (Riemann) integrable on $[a, b]$ then

$$\int_a^b f(x) g'(x) dx = f(b) g(b) - f(a) g(a) - \int_a^b f'(x) g(x) dx.$$

When stated in this way, the theorem is easy to use and its proof is suggested simply and precisely by the product rule

$$f g' = (f g)' - f' g.$$

The Integration by Parts Theorem as Stated in the Calculus Books

The equation

$$\int_a^b u dv = u(b)v(b) - u(a)v(a) - \int_a^b v du$$

holds whenever u and v are “appropriate” functions defined on a given interval $[a, b]$.

Actually, for practical purposes, the functions u and v might be taken to be continuous functions of bounded variation. To be honest, I should say that the continuity isn’t really needed; but, without continuity, the theorem would have to be stated much more carefully.¹ The integrals in the equation

$$\int_a^b u dv = u(b)v(b) - u(a)v(a) - \int_a^b v du$$

have to be interpreted as **Riemann-Stieltjes** integrals.

My Objection

On about a zillion occasions, when I have objected to the phrasing of a simple elementary integration technique in the language of much more advanced mathematics, I have been told that I am fussing; because integrals such as

$$\int_a^b u dv$$

allegedly, aren’t actually Stieltjes integrals at all; because (so I am told) what we really mean by the latter expression is

$$\int_a^b u(x) v'(x) dx.$$

¹ On request, I can provide a detailed account of the full force of the integration by parts principle for Riemann-Stieltjes making use of the Kenneth Ross version of Riemann-Stieltjes integrability. This theorem does not appear in any calculus or analysis texts that are presently available.

In other words, I am informed that the expression

$$\int_a^b u dv$$

is nothing more than the integral

$$\int_a^b u(x) v'(x) dx.$$

in which a change of variable is made from x to v .

However, people who make the latter claim are wrong, wrong, wrong. In my opinion, they have not understood the subject matter and need to do some hard rethinking in order to be competent teachers of calculus. The equation

$$\int_a^b u dv = \int_a^b u(x) v'(x) dx$$

has **nothing whatsoever** to do with the change of variable principle. Instead, the equation is an instance of the theorem on reduction of Riemann-Stieltjes integrals to Riemann integrals:

Suppose that ϕ is an increasing function whose derivative ϕ' is Riemann integrable on an interval $[a, b]$ and suppose that f is Riemann-Stieltjes integrable with respect to ϕ on $[a, b]$. Then the function $f \phi'$ is Riemann integrable on the interval $[a, b]$ and we have

$$\int_a^b f d\phi = \int_a^b f(x) \phi'(x) dx.$$

In sharp contrast to this reduction theorem, the change of variable theorem says:

Suppose that u is a differentiable function on an interval $[a, b]$ and that its derivative u' is integrable on $[a, b]$. Then given any function f that is integrable on the range of u we have

$$\int_a^b f(u(t)) u'(t) dt = \int_{u(a)}^{u(b)} f(x) dx.$$

Apart from the superficial similarity between these statements that comes

from the notion that

$$dv = v'(x) dx$$

in the first theorem and that

$$u'(t) dt = du$$

in the second theorem, the two statements play completely different roles.

Even though the theorem on reduction of Riemann-Stieltjes integrals to Riemann integrals is much easier to prove, it is, at the very least, an advanced fact. It depends upon the sophisticated notion of a Stieltjes integral. The change of variable theorem, by contrast, though excruciatingly difficult to prove (as stated above), deals with elementary topics. Note how the formula

$$\int_a^b f(u(t)) u'(t) dt = \int_{u(a)}^{u(b)} f(x) dx$$

works with the integral of the composition of the functions f and u times u' on the left and note how this theorem requires a careful change in the limits of integration. There is no such change in the limits of integration in the theorem on reduction of a Riemann-Stieltjes integral to a Riemann integral and there is no change in the limits of integration in the integration by parts formula.

It is difficult enough to train students to use the change of variable method correctly and to change the limits of integration as they do so.... Do we really want to subject them to what they (and, unfortunately, many instructors) see as the same process in which the limits of integration suddenly have to be left unchanged??????????????

Voodoo mathematics indeed.

I can hardly think of a more efficient way to steer our students into a hatred and mistrust of mathematics and of their instructors, than the blatant misuse of notion that I have described here.

I suggest that the world would not come to an end if we applied integration

by parts by saying, for example, that

$$\begin{aligned}\int_0^\pi x \cos x dx &= \int_0^\pi (x) \left(\frac{d}{dx} \sin x \right) dx \\ &= \left[(x) (\sin x) \right]_0^\pi - \int_0^\pi \left(\frac{d}{dx} x \right) (\sin x) dx \\ &= 0 - 0 - \int_0^\pi (1) (\sin x) dx = -2\end{aligned}$$