Closed Knight's Tours with Minimum Square Removal for all Rectangular Boards

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A closed knight's tour of a chessboard is a classic problem in mathematics. Can a knight use legal moves to visit every square on the board and return to its starting position? An open knight's tour is a knight's tour of every square that does not return to its starting position. While originally studied for the standard 8×8 board, the problem is easily generalized to other rectangular boards. In 1991 Schwenk classified all rectangular boards that admit a closed knight's tour.

Schwenk's Theorem: An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of the following three conditions hold:

(a) m and n are both odd;
(b) $m \in \{1, 2, 4\}$;
(c) $m = 3$ and $n \in \{4, 6, 8\}$.

How close to admitting a closed knight's tour are those boards that satisfy the conditions of Schwenk's Theorem? There is no closed knight's tour of the 3×3 board. However, once the center square is removed a closed knight's tour does indeed exist as seen in Figure 1.

1	6	3
4		8
7	2	5

Figure 1

Let the *tour number*, $T(m, n)$, with $m \leq n$ be the minimum number of squares whose removal from an $m \times n$ chessboard will allow a closed knight's tour. Thus, $T(3, 3) = 1$. Note that for m and n that do not satisfy the conditions of Schwenk's Theorem, $T(m, n) = 0$. Also note that the removal of a random tour number of squares does not guarantee the existence of a closed knight's tour. Removing any square other than the center does not allow for a closed knight's tour of the 3×3 board. Furthermore $T(1, n)$ and $T(2, 2)$ are undefined since the knight cannot move from its starting position. Also $T(2, n) = 2n - 2$ for $n \geq 3$ since a knight can move down a $2 \times n$ board but cannot return to its starting position unless only one move has been made.

There are restrictions on the number of squares that can be removed and potentially allow a closed knight's tour to exist. Throughout this paper, whenever we color the squares of a chessboard black and white, we will always begin with a black square in the upper left-hand corner. A legal move for a knight whose initial position is a white square will always result in an ending position on a black square and vice versa. Hence, any closed knight's tour must visit an equal number of black squares and white squares. This quickly determines that an odd number of squares must be removed from a board where both m and n are odd and an even number of squares must be removed from all other boards. For a board that does satisfy the conditions of Schwenk's Theorem, the smallest possible tour numbers are 1 and 2 respectively for boards with an odd or even number of squares.

The Case of $m = 3$

a	d	g	j
l	i	b	e
c	f	k	h

Figure 2

To construct a closed knight's tour for the 3×7 board, start with the tour for the 3×3 board with the center square removed as in Figure 1. Delete the $5 - 6$ edge. Next, take the open 3×4 tour from Figure 2 and create the $5 - a$ and $6 - l$ edges which correspond to legal moves of a knight as shown in Figure 3.

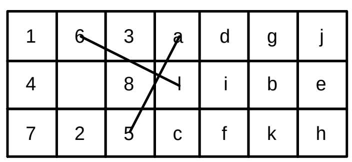

Figure 3

This game can be played an infinite number of times replacing the role of vertices 5 and 6 by g and h. Thus, $T(3, n) = 1$ for all $n \equiv 3 \mod 4$. Note that the lower right hand corner of any board, when used, must contain the $g - h$ edge as there are only two legal moves for a knight from that corner square.

The case of the 3×4 board is a very straightforward one. As shown in Figure 4, if two vertices are removed, a knight's tour exists. Thus, $T(3, 4) = 2$. Furthermore, using the open tour of Figure 2 shows that $T(3, 8) = 2$. Since $T(3, n) = 0$ for all even $n \geq 10$ no additional tours will be constructed from this case.

1	4	9	6
	7	2	
3	10	5	8

Figure 4

For the 3×5 board note that the corner squares will be vertices of degree two in the corresponding graph and are all adjacent to the center square. Thus, if any of those squares are included in a closed knight's tour they must be adjacent to the center square. Clearly, at most two of these four vertices may be included. Hence at least two squares must be removed. Furthermore, all four of those corner squares are black and it will be necessary to remove at least one white square. Thus, $T(3,5) \geq 3$. The existence of the tour in Figure 5 shows that $T(3,5) = 3$.

1	4	7	10	
	9	12	3	6
	2	5	8	11

Figure 5

But the 3×5 board is the lone exception for all boards with an odd number of squares. Figure 6 shows that $T(3,9) = 1$. Once again, using the open tour of Figure 2 yields $T(3,n) = 1$ for all $n \equiv 1 \mod 4$ where $n \neq 5$.

1	4	7	18	21	24	9	12	15
6	19	2	25	8	17	14	23	10
3	26	5	20		22	11	16	13

Figure 6

All that is left is the 3×6 board and analysis will show $T(3, 6) = 4$.

1	4	7	10	13	16
2	5	8	11	14	17
3	6	9	12	15	18

Figure 7

Using all four corners immediately forces the paths $6 - 1 - 8 - 3 - 4$ and $13 - 18 - 11 - 16 - 15$. Since vertices 8 and 11 can have no further adjacencies these paths are necessarily extended to $7 - 6 - 1 - 8 - 3 - 4 - 9$ and $12 - 13 - 18 - 11 - 16 - 15 - 10$. However none of the four remaining vertices can be included without closing one of these paths before connecting to the other. No tour exists using all four corners that omits exactly two squares. Furthermore, including the $7 - 12$ and $9 - 19$ moves creates a tour when omitting 4 squares and $T(3, 6) \leq 4$.

Next note that squares 5 and 17 cannot both be used without creating a closed cycle with 10 and 12. Similarly, squares 2 and 14 cannot both be used without creating a closed cycle with 7 and 9. When combined with the previous fact that no tour exists using all four corners that omits exactly two squares we achieve $T(3, 6) \geq 3$. Since $T(3, 6)$ is even, $T(3, 6) = 4$.

The Case of $m = 4$ With $m = 4$, the board will have an even number of squares. Thus, $T(4, n)$ will always be even. The boards of Figure 8 show that $T(4, 4) = T(4, 5) = T(4, 6) = 2$.

	5	10	1
13	2	7	4
6	9	14	11
	12	3	8

12	3	16	7	10
17	8	11	2	15
4	13	18	9	6
		5	14	1

21	2	13	8	17	4
12	7	22	3	14	9
1	20	11	16	5	18
		6	19	10	15

Figure 8

Next consider any $4 \times k$ tour that contains the $a - b$ and $c - d$ edges in the lower right-hand corner as in Figure 9. This $4 \times k$ tour can now be extended to a $4 \times (k+3)$ tour by first removing edges $a - b$ and $c - d$. Next include the edges $1 - c$, $6 - d$, $a - a_1$, $b - a_6$. Note that all three base boards and the 4×3 extension contain the $a - b$ and $c - d$ edges in the lower right-hand corner as in Figure 9. This proves $T(4, n) = 2$ for all $n \geq 4$.

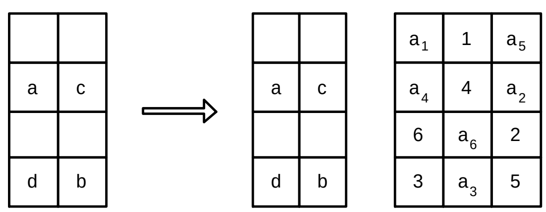

Figure 9

The Case of Both m and n odd Much like for $m = 3, 4$, induction with

an appropriate base case will be used to construct all boards with an odd number of squares for $m \geq 5$. In all cases, $T(m, n) = 1$ for both m and n odd with $5 \leq m \leq n$. Four base cases exist, one for each combination of $m, n \equiv 1, 3 \mod 4$. The boards of Figure 10 are used for $m, n \equiv 1 \mod 4$ and $m \equiv 1, n \equiv 3 \mod 4$ respectively. For $m, n \equiv 3 \mod 4$ use the 3×7 board of Figure 3 and for $m \equiv 3, n \equiv 1 \mod 4$ use the 3×9 board of Figure

6. The open 3×4 tour of Figure 2 and the open 5×4 tour of Figure 11 can be used to extend the base boards to any length $n \equiv 1, 3 \mod 4$ as demonstrated in Figure 3.

1	18	7	12	
8	13	24	17	22
19	2	21	6	11
14	9	4	23	16
3	20	15	10	5

1	30	17	8	23	28	15
18	9	34	29	16	7	24
31	2		22	25	14	27
10	19	4	33	12	21	6
3	32	11	20	5	26	13

Figure 10: Base Boards for $m \equiv 1 \mod 4$

12	17	8	3
7	2	13	18
16	11	4	9
1	6	19	14
20	15	10	5

Figure 11:
Open 5×4
Tour

We have constructed tours for all $3 \times n$ and $5 \times n$ boards for odd $n \geq 7$. Next we need to extend these boards down to an arbitrary odd m. To do so, rotate clockwise the open tours of Figure 2 and Figure 11 to a 4×3 tour and a 4×5 tour and extend the base $3 \times n$ and $5 \times n$ boards down to any depth $m \equiv 1, 3 \mod 4$, again, as demonstrated in Figure 3. For $n \equiv 1 \mod 4$ use the rotated clockwise 4×5 board of Figure 11. For $n \equiv 3 \mod 4$ use the rotated clockwise 4×3 board of Figure 2. This process provides us with a closed knight's tour of the top and left side of the board in Figure 12.

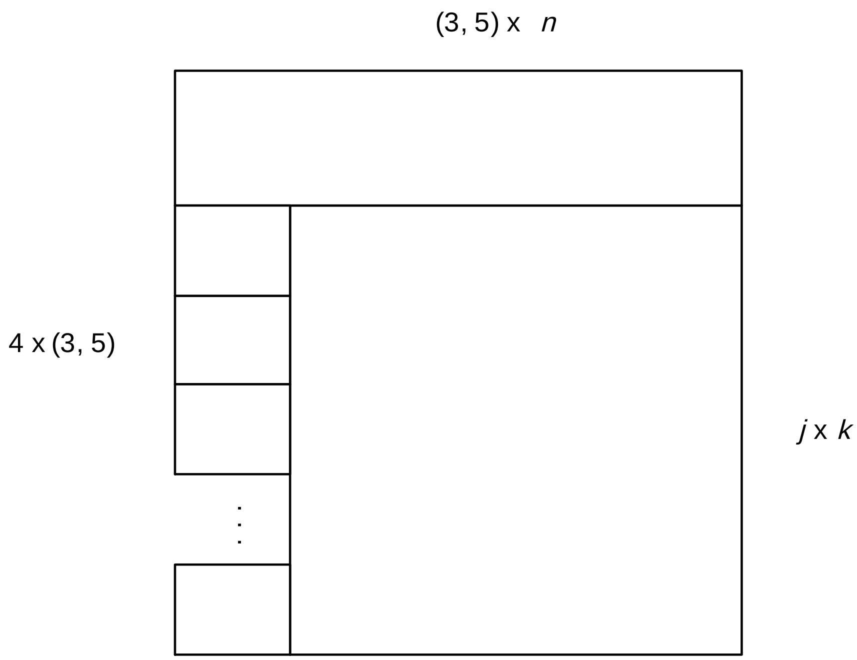

Figure 12

Now a $j \times k$ gap with $j, k \equiv 0 \mod 4$ needs to be filled in to complete the $m \times n$ board. Finally, we use the 4×4 board of Figure 13 to fill in the $j \times k$ gap using the same technique of Figure 9.

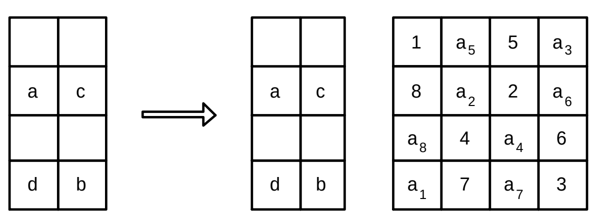

Figure 13

Conclusion In summary, the tour number for a board that does satisfy the conditions of Schwenk's Theorem is as small as possible (1 or 2) based on an odd or even number of squares with the few noted exceptions as indicated below.

For the $m \times n$ chessboard with $m \leq n$, either $T(m, n) = 0$ and the board has a closed knight's tour, or else

(a) $T(m, n) = 1$, where m and n are both odd except for $m = 3$ and $n = 5$;
(b) $T(4, n) = 2$ for all $n \geq 4$;
(c) $T(3, 4) = T(3, 8) = 2$, $T(3, 5) = 3$, $T(3, 6) = 4$;
(d) $T(2, n) = 2n - 2$ for $n \geq 3$;
(e) $T(1, n)$ and $T(2, 2)$ are undefined.

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