5.3 The Pigeonhole Principle

The basic form of the pigeonhole principle is that if $k+1$ pigeons are placed into k pigeonholes then at least one pigeonhole will contain at least two pigeons.

Figure 5.3.1

Clearly, if all three of the pigeons of Figure 5.3.1 are placed into the two pigeonholes then one hole will contain at least two pigeons. It could be the case that all three pigeons are placed into one hole so it is not appropriate to say that one pigeonhole will contain exactly two pigeons. If five pigeons were placed into four pigeonholes then at least one hole will contain at least two pigeons. It certainly could be the case that three pigeonholes contain exactly one pigeon each and the fourth contains two pigeons. However, it could also be the case that two pigeonholes contain two pigeons each, a third contains one pigeon and the remaining two pigeonholes are empty. Or once again, all five pigeons could be placed into one pigeonhole, leaving the other four pigeonholes empty.

The pigeonhole principle can be used to show that in every graph with at least two vertices there are at least two vertices of the same degree. Let $G=(V,E)$ be a graph with $n \geq 2$ vertices. The maximum degree in such a graph is $n-1$. However, there can not exist a vertex of degree 0 and another vertex of degree $n-1$ in a graph of n vertices. Thus, there are $n-1$ possible degrees (either $0,1,2,...,n-2$ or $1,2,3,...,n-1$) to distribute among n vertices. Thus, at least two vertices have the same degree in any graph with $n \geq 2$ vertices.

An extended form of the pigeonhole principle is that if n pigeons are placed into k pigeonholes, then at least one pigeonhole will contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons where the notation represents the smallest integer greater than $\frac{n}{k}$. This is sometimes called the ceiling function. If 10 pigeons are placed into 3 pigeonholes then at least one pigeonhole will contain at least $\left\lceil \frac{10}{3} \right\rceil = 4$ pigeons. The same result will be true if 11 or 12 pigeons are placed into 3 pigeonholes. However, once 13 pigeons are placed into 3 pigeonholes then at least one pigeonhole will contain at least $\left\lceil \frac{13}{3} \right\rceil = 5$ pigeons.

For example, if Michael Jordan scored 57 points in four regulation quarters of a basketball game then there was at least one quarter where he scored at least $\left\lceil \frac{57}{4} \right\rceil = 15$ points.

Let $S = \{x_1, x_2, ..., x_{42}\}$ where x_i is an integer $1 \le x_i \le 70$ and $x_i \ne x_j$ for $i \ne j$. Prove there exists i and j such that $x_i - x_j = 13$. For $x_i = n$, place x_i into pigeonhole n. Construct 42 new objects x_i^*. For $x_i = n$, place x_i^* into pigeonhole number $n + 13$. Thus, $42 + 42 = 84$ different pigeons have been placed. The smallest possible pigeonhole used is numbered 1 if the smallest possible pigeon $x_1 = 1$. If the largest possible pigeon $x_{42} = 70$, then the largest possible pigeonhole used is numbered 83 for placing x_{42}^*. Hence, there are 83 different pigeonholes. By the pigeonhole principle, 84 pigeons have been placed into 83 pigeonholes and at least one pigeonhole contains at least two pigeons. Can the pigeons x_i and x_j be in the same pigeonhole for $i \ne j$? No. If so, then $x_i = x_j$ for $i \ne j$, which is a contradiction. Similarly, can the pigeons x_i^* and x_j^* be in the same pigeonhole for $i \ne j$? For the same reason, no. Clearly by the placement of the pigeons, x_i and x_i^*cannot be in the same pigeonhole. Thus, it is the case that x_i and x_j^* must be in the same pigeonhole for some $i \ne j$. Thus, $x_i = x_j + 13$ and $x_i - x_j = 13$.

Homework

1. Does a class of 12 students have at least 2 students who share the same birth month?
   Does a class of 13 students have at least 2 students who share the same birth month?

2. Does a class of 25 students have at least 2 students who share the same birth month?
   Does a class of 25 students have at least 3 students who share the same birth month?
   Does a class of 25 students have at least 4 students who share the same birth month?

3. At a university of 13,000 students, at least how many must share the same birthday (not including the year)? Don't forget leap years. How many must have the birthday September 19th?

4. At a university of 13,000 students, at least how many must share the same 4-digit pin number for their ATM card (assuming that each student has an ATM card)?

5. John has 13 ordinary coins (pennies, nickels, dimes and quarters) in his pocket. At least how many of the same coin must John have? At least how many pennies must John have?

6. A cleaning service has 46 homes to clean in February. At least how many homes were cleaned in the first week? At least how many homes were cleaned within some seven day period, assuming that this is not a leap year?

7. Let $S = \{x_1, x_2, ..., x_{58}\}$ where x_i is an integer, $1 \le x_i \le 82$ and $x_i \ne x_j$ for $i \ne j$. Prove there exist i and j such that $x_i - x_j = 33$.

8. Let $S = \{x_1, x_2, ..., x_{47}\}$ where x_i is an integer, $1 \le x_i \le 75$ and $x_i \ne x_j$ for $i \ne j$. Prove there exist integers i and j such that $x_i - x_j = 18$.

9. A cleaning service has 46 homes to clean in 30 days and will clean at least one house a day. Show that there is a period of consecutive days where exactly 13 homes are cleaned.

10. In a set of 6 people, show that there exists a subset of at least 3 people who either all know every other person in the subset or there exists a subset of at least 3 people all of whom are strangers to one another. You may assume that if person *A* knows person *B*, then person *B* also knows person *A*. Show that this result is not true for a set of 5 people.