1 Induction

The ancient puzzle, The Towers of Hanoi, consists of three pegs and n rings of different sizes. The n rings are placed on one peg with the rings ordered top to bottom from smallest to largest. A move in this puzzle is performed by selecting any top ring from any peg and moving it to the top of one of the other pegs. It is not permissible to place a larger ring on top of a smaller ring. Can all the rings be moved from the starting peg to either of the other two pegs? If so, how many moves are required to do so?

$1 \rightarrow b$
$2 \rightarrow c$
$1 \rightarrow c$

The pegs are labeled a, b and c from left to right. We will use the convention of consecutively numbering the rings from 1 to n where 1 is the smallest ring and all rings start on peg a. Obviously, for $n = 1$ rings the minimum number of moves is one. A solution for $n = 2$ rings is simple to construct. Consider the sequence of moves shown in the chart. First move ring 1 to peg b, then move ring 2 to peg c, and finally move ring 1 to peg c.

$1 \rightarrow b$
$2 \rightarrow c$
$1 \rightarrow c$
$3 \rightarrow b$
$1 \rightarrow a$
$2 \rightarrow b$
$1 \rightarrow b$

A solution for $n = 3$ rings requires a bit more effort, but not a Herculean one. The chart to the left shows the seven moves in the sequence.

A critical observation at this juncture is to note that the solution for $n = 2$ rings is the first series of three moves and (changing the labeling of the pegs) the last series of three moves of the solution for $n = 3$ rings. The only move that has nothing to do with the solution for $n = 2$ rings is the middle move of the bottom ring to the middle peg. This is not a coincidence. The solution for $n = 4$ rings is to use the solution for $n = 3$ rings to move rings 1, 2 and 3 from peg a to peg c, move ring 4 from peg a to peg b and use the solution for $n = 3$ rings to move rings 1, 2 and 3 from peg c to peg b. It initially appears that $2^n - 1$ moves are required for a solution to the Towers of Hanoi problem with n rings. Can this formula be proven true for every value of n? If so, how?

Mathematical Induction is a very powerful tool when proving theorems about the integers. Intuitively, one should visualize induction as setting up an infinite number of dominoes. These dominoes are positioned in such a way so that a falling domino always knocks the next one down. When the first domino is tipped over, a chain of falling dominoes ensues and they all fall down. In proving theorems, the dominoes represent the integers. The approach to a proof that the statement $S(k)$ is true for the integers is twofold:

Step One: Show that $S(k)$ is true for some initial integer n. This is the base case.

Step Two: Prove that if $S(k)$ is true for all integers k less than or equal to n then the $S(n+1)$ is true.

Step Two sets the dominoes up in such a way as to always knock the next one down. Step One tips the first domino so that they all fall. The statement "if the theorem is true for all integers less than or equal to n" is called the induction hypothesis.

Using induction, it is easy to prove that a solution to the Towers of Hanoi problem with n rings uses $2^n - 1$ moves. A base case for $n = 2$ has already been established. The solution requires three moves. Since $2^2 - 1 = 3$, Step One is satisfied. Next, assume that a solution exists for n rings with $2^n - 1$

moves. The goal is to show that a solution for $n+1$ rings using $2^{n+1}-1$ moves exists. Beginning with the rings on peg a, use the solution for n rings to move rings 1 though n to peg c using 2^n-1 moves. With a single move, place ring $n+1$ on peg b. Finally, using the solution for n rings again, move rings 1 through n to peg b using 2^n-1 moves. This is clearly a solution for $n+1$ rings that uses $2^n-1+1+2^n-1=2*2^n-1=2^{n+1}-1$ moves and Step Two is satisfied. Hence, by the principle of mathematical induction there exists a solution using 2^n-1 moves to the Towers of Hanoi problem with n rings for all $n \geq 2$. If a proof for $n=1$ is required, the only necessary change is to prove that the base case holds for $n=1$.

Summation formulas often come from counting techniques and can be proven using induction. Consider the sum of the first n integers. Induction can be used to show that the sum of the first n integers is $\frac{n(n+1)}{2}$.

Theorem 1 $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$

Proof. First the base case $n=1$ must be shown to be true. For $n=1$, the left hand side of the equality is $\sum\limits_{k=1}^{1} k = 1$. On the right hand side, $\frac{1*2}{2}=1$. The equality holds for the base case $n=1$. For the second part, assume the equality holds for the integer n (i.e. $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$) and show that it holds for the integer $n+1$ (i.e. $\sum\limits_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$).

$\sum\limits_{k=1}^{n+1} k = \left(\sum\limits_{k=1}^{n} k\right) + (n+1)$

$= \frac{n(n+1)}{2} + (n+1)$ by the inductive assumnption

$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$

$= \frac{n^2+3n+22}{2}$

$= \frac{(n+1)(n+2)}{2}$

This demonstrates that if the statement of the theorem is true for n then the statement of the theorem is true for $n+1$. Both parts of the mathematical induction have been satisfied and the equality holds for all positive integers n. ■

Removing the last term of the summation $\sum\limits_{k=1}^{n+1} k$ allows the proof to take advantage of the induction hypothesis. In general, the key to a successful induction proof is to find a way to use the induction hypothesis. There are other ways to prove $\sum\limits_{k=1}^{n} k = \frac{n(n+1)}{2}$ but it also provides an excellent illustration of induction.

Theorem 2 *For* $n \in \mathbb{Z}^+$, $\frac{4^{2n+1}-25}{3}$ *is an integer.*

Proof. *As is almost always true, proving the base case is simple. For* $n=1$, *note that* $\frac{4^{2(1)+1}-25}{3} = \frac{4^3-25}{3} = \frac{39}{3} = 13$ *is an integer. Next, assume the state-*

ment is true for n (i.e.. $\frac{4^{2n+1}-25}{3}$ *is an integer) and show that the statement is true for* $n+1$ *(i.e..* $\frac{4^{2(n+1)+1}-25}{3} = \frac{4^{2n+3}-25}{3}$ *is an integer. In order to use the induction hypothesis the form* $\frac{4^{2n+1}-25}{3}$ *must be found in* $\frac{4^{2n+3}-25}{3}$*. This requires some strategic algebraic manipulation.*

$\frac{4^{2n+3}-25}{3} = \frac{4^2*4^{2n+1}-25}{3}$

$= \frac{(15+1)*4^{2n+1}-25}{3}$

$= \frac{15*4^{2n+1}}{3} + \frac{4^{2n+1}-25}{3}$

$= 5*4^{2n+1} + \frac{4^{2n+1}-25}{3}$

Clearly $5*4^{2n+1}$ *is an integer and* $\frac{4^{2n+1}-25}{3}$ *is an integer by the inductive hypothesis. The sum of any two integers is also an integer. We have shown that if* $\frac{4^{2n+1}-25}{3}$ *is an integer then* $\frac{4^{2n+3}-25}{3}$ *is also an integer. Both parts of the mathematical induction have been satisfied and for* $n \in \mathbb{Z}^+$, $\frac{4^{2n+1}-25}{3}$ *is an integer.* ■

As can be seen creative manipulations are sometimes necessary to use the induction hypothesis. It is important to focus on breaking the form of the inductive hypothesis out in Step Two.

Theorem 3 *For integers* $n \geq 5$, $(n+1)! > 5^{n-1}$.
Proof. *For* $n = 5$, $6! = 720 > 625 = 5^4$ *and the base case for* $n = 5$ *is true. For Step Two, assume the statement is true for n (i.e.* $(n+1)! > 5^{n-1}$*) and show the statement is true for* $n+1$ *(i.e.* $(n+2)! > 5^n$*). Note that* $(n+2)! = (n+2)(n+1)! > (n+2)5^{n-1}$ *by the inductive assumption. Since* $n \geq 5$ *implies* $n+2 > 5$, *it must be true that* $(n+2)5^{n-1} > 5*5^{n-1} = 5^n$*. Both parts of the mathematical induction have been satisfied and* $(n+1)! > 5^{n-1}$ *for* $n \geq 5$. ■

n	$p(n)$
0	$p(0) = 41$
1	$p(1) = 43$
2	$p(2) = 47$
3	$p(3) = 53$
4	$p(4) = 61$
5	$p(5) = 71$

Many students experience discomfort when first studying induction because it seems possible to prove just about any pattern that holds true for just a couple of cases. But this is not so. For example, consider the function $p(n) = n^2 + n + 41$. The first few non-negative integer values of n yield prime outputs for $p(n)$. Can it now be shown using induction that $p(n)$ generates a prime number for every non-negative integer value n?

The base case $n = 0$ is true because $p(0) = 41$ is prime. Next, assume that $p(n)$ is a prime number and attempt to show that $p(n+1)$ is also prime. Evaluating the function at $n+1$ and simplifying gives the following.

$$p(n+1) = (n+1)^2 + (n+1) + 41 = n^2 + n + 41 + 2n + 2 = p(n) + 2n + 2$$

So the induction hypothesis may be used on $p(n)$, just as has happened in previous proofs. Let's specifically take a look at the proof of Theorem 2. In that proof, the argument is reduced down by the fact that the inductive hypothesis assumes an expression to be an integer. This integer expression is then added to another integer. Of course, the sum of any two integers is another integer and the result is attained. However, in this problem the inductive hypothesis is that $p(n)$ is prime. But, there is no subsequent result that shows that a prime number added to $2n+2$ results in another prime number. So, it is not possible to show that if the statement is true for the value of n then the statement is true for the value of $n+1$. There is no way to use the induction assumption to prove that the result holds for the next integer value.

The original question still remains. Does the polynomial expression $p(n)$ generate prime numbers for all non-negative integers? Although there appears to be a pattern of prime numbers for a long time, it does not hold for all non-negative integers. Take note that $p(41) = 1763 = 41 \times 43$ and is composite.

Occasionally there is more work in the base case itself than in the inductive step. In fact, a problem may require multiple base cases to get the induction started. If only 4-cent and 5-cent postage stamps are available show that any postage amount greater than 11 cents can be made. After constructing different cases for 12 cents, 13 cents, 14 cents and 15 cents, we see that any postage amount greater than 15 cents reduces down to one of these four base cases plus some number of 4 cent stamps. While it is rare that more work goes into the base cases than the inductive step, this does serve as a nice reinforcement of the importance of the base case(s).

$$\begin{gathered} 12 = 3 * 4 \\ 13 = 2 * 4 + 5 \\ 14 = 4 + 2 * 5 \\ 15 = 3 * 5 \end{gathered}$$

Exercise 1 *Prove* $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ *for* $n \in \mathbb{Z}^+$.

Exercise 2 *Prove* $\sum_{k=0}^{n} k^3 = \frac{n^2(n+1)^2}{4}$ *for* $n \in \mathbb{Z}^+$.

Exercise 3 *Prove* $\sum_{k=0}^{n} (2k-1) = n^2$ *for* $n \in \mathbb{Z}^+$.

Exercise 4 *Prove* $\sum_{k=1}^{n} k * k! = (n+1)! - 1$.

Exercise 5 *Prove* $\sum_{k=1}^{n} k2^k = 2 + (n-1)\, 2^{n+1}$.

Exercise 6 *Find and prove the correctness of a formula for* $\sum_{k=0}^{n} 2k$ *for* $n \in \mathbb{Z}^+$.

Exercise 7 *Show that is* $\frac{(2n)!}{2^n}$ *an integer for* $n \in \mathbb{Z}^+$.

Exercise 8 *Show that is* $\frac{(3n)!}{6^n}$ *an integer for* $n \in \mathbb{Z}^+$.

Exercise 9 *Show that is* $\frac{n^3-n+12}{6}$ *an integer for* $n \in \mathbb{Z}^+$.

Exercise 10 *Show that is* $\frac{2^{3n}-22}{7}$ *an integer for* $n \in \mathbb{Z}^+$.

Exercise 11 *Show that is* $\frac{2^{3n}+3^{n+2}}{5}$ *an integer for* $n \in \mathbb{Z}^+$.

Exercise 12 *Show that is* $\frac{5^{n+2}+9^n+10}{4}$ *an integer for* $n \in \mathbb{Z}^+$.

Exercise 13 *Show that* $n! \geq 2^n$ *for integers* $n \geq 4$.

Exercise 14 *Show that* $(2n)! > 3^{2n+1}$ *for integers* $n \geq 4$.

Exercise 15 *If only 3-cent and 5-cent postage stamps are available show that any postage amount greater than 7 cents can be made.*

Exercise 16 *If only 5-cent and 6-cent postage stamps are available show that any postage amount greater than 19 cents can be made.*