5.2 Arrangements with Forbidden Positions

A **derangement** is a permutation of the first n integers such that the element i is not in the i^{th} position for all $i \le n$. If i is in the i^{th} position, we say it is in its **natural position**. The permutation 32514 is not a derangement because 2 is in its natural position. However, 53124 is a derangement since no i is in the i^{th} position. No derangements exist on the single digit 1. There is only a single derangement of the first two integers, namely 21. For $n = 3$, there exist two different derangements, 231 and 312. For ease of reference, d_i represents the number of derangements of the first n integers. Thus, $d_1 = 0$, $d_2 = 1$ and $d_3 = 2$.

Theorem 5.2.1: For $n \ge 2$, $d_n = n! \sum_{i=2}^{n} \frac{(-1)^i}{i!}$.

Proof: Clearly, d_n will be equal to the total number of permutations of n objects once the permutations that contain at least one digit in its natural position are removed. Let S be the set of permutations on n objects that contain at least one integer in its natural position. Hence, $d_n = n! - |S|$. Let A_i be the subset of S that contains the permutations with i in its natural position. By the principle of inclusion/exclusion,
$|S| = \sum_i |A_i| - \sum_{\substack{i,j \\ i \ne j}} |A_i \cap A_j| + -... + -|A_1 \cap A_2 \cap ... \cap A_n|$ For all i, $|A_i| = (n-1)!$ and $\sum_i |A_i| = \binom{n}{1}(n-1)!$. Similarly, $|A_i \cap A_j| = (n-2)!$ and $\sum_{\substack{i,j \\ i \ne j}} |A_i \cap A_j| = \binom{n}{2}(n-2)!$. In general, the sum of the intersection of k sets is $\binom{n}{k}(n-k)!$. Next, note that $\binom{n}{k}(n-k)! = \frac{n!}{k!(n-k)!}(n-k)! = \frac{n!}{k!}$. Hence, $|S| = \sum_{i=1}^{n}(-1)^{i-1}\frac{n!}{i!} = n! \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i!}$. Finally, $d_n = n! - |S| = n! - n! \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i!} = n! \sum_{i=2}^{n} \frac{(-1)^i}{i!}$. ❑

Thus, by Theorem 5.2.1, $d_4 = 4!(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}) = 24(\frac{9}{24}) = 9$.

Theorem 5.2.2: For $n \ge 3$, $d_n = (n-1)(d_{n-1} + d_{n-2})$.

Proof: Consider the derangements on the first n integers. In the derangement, the first element i can be any value other than 1. Thus, there are $n-1$ choices for the value i. There are now two cases to consider. Either the i^{th} position is 1 or it is not. If 1 is in the i^{th} position then there are $n-2$ elements left to be arranged as a derangement. Clearly, there are d_{n-2} ways to finish such a derangement. If 1 does not appear in the i^{th} position then re-label 1 as i. We now have the elements $2, 3, ..., i, ..., n$ to permute as a derangement. Remember that i represents 1. This can be done in d_{n-1} ways. Re-label i as 1 and we have the remaining derangements where 1 is not in the i^{th} position. Thus, $d_n = (n-1)(d_{n-1} + d_{n-2})$. ❑

Thus, by Theorem 5.2.2, $d_4 = 3(d_3 + d_2) = 3(2+1) = 9$.

A class of 15 students are to critique an in-class writing assignment of their fellow students. How many different ways can the papers be distributed such that no one critiques their own assignment? Clearly, every such distribution is a derangement and $d_{15} = 481,066,515,734$.

A derangement is a special case of a problem where specific entries are forbidden. The next goal is to generalize the problem in order to be able to exclude any position and not just one specific position. Consider the assignment of different math courses to be taught by faculty. Certain faculty have different areas of expertise and preferences for their courses. The table in Figure 5.2.1 represents a preference with a ✓ and a non-preference with a ✗.

	Math 1190	Math 2202	Math 3332	Math 3333
Burke	✓	✓	✗	✗
Gooch	✗	✗	✓	✓
Laval	✗	✓	✓	✓
Watson	✓	✓	✓	✗

Figure 5.2.1

A solution to this problem requires selecting exactly one ✓ from each row and column. This is equivalent to placing four non-taking rooks on the acceptable white spaces of the chessboard in Figure 5.2.2.

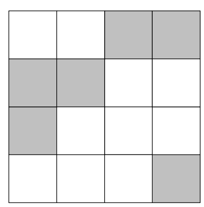


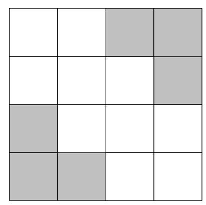
Figure 5.2.2.

The number of ways to place four non-taking rooks on the acceptable white spaces is 4! minus the number of ways to place four non-taking rooks using at least one of the unacceptable shaded spaces. Let r_i denote the number of ways to place at least i non-taking rooks on shaded squares. As there are six shaded squares on the board, $r_1 = 6$. For any of those six arrangements the rest of the non-taking rooks can be arranged in 3! ways for 36 arrangements. Placing two non-taking rooks can be done in eleven ways, $r_2 = 11$ and there exist $11 * 2! = 22$ such arrangements. Three non-taking rooks can be placed in $r_3 = 6$ ways. Finally, there is only one way to place four non-taking rooks on shaded squares and $r_4 = 1$. As you can see computing r_i is a nontrivial task. Once, computed the above sum becomes simple to compute. Thus, the number of acceptable solutions is $4! - r_1 * 3! + r_2 * 2! - r_3 * 1! + r_4 * 0! = 5$.

For any $n \times n$ board B, the **rook polynomial** for B, denoted $R(B, x)$, is the polynomial where the coefficient of x^i is r_i. As a matter of convenience, define $r_0 = 1$. The rook polynomial of the board in Figure 5.2.2 is $x^4 + 6x^3 + 11x^2 + 6x + 1$. Given an $n \times n$ board and its rook polynomial $R(B,x)$, there are $\sum_{i=0}^{n}(-1)^i r_i(n-i)!$ different arrangements of n non-taking rooks on non-shaded squares. Two sub-boards B_1 and B_2 are said to be disjoint if no rows or columns share forbidden squares.

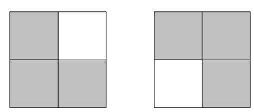
Theorem 5.2.3: If a board B is broken into two disjoint boards B_1 and B_2 then $R(B,x) = R(B_1,x) * R(B_2,x)$.

By exchanging rows 1 and 3 of the board in Figure 5.2.2, the board B in Figure 5.2.3 is created. It is then easy to decompose this board B into the two disjoint boards of Figure 5.3.4.



B

Figure 5.2.3



B_1 B_2

Figure 5.2.4

Computing the rook polynomials of these two smaller boards is a much simpler task than determining the rook polynomial of the original board. With a little work we find that $R(B_1,x) = R(B_2,x) = 1 + 3x + x^2$. Hence,
$R(B,x) = R(B_1,x) * R(B_2,x) = (1 + 3x + x^2)^2 = x^4 + 6x^3 + 11x^2 + 6x + 1$.

Consider the case where five children must select one of five different snacks. Of course, not every child desires every snack. The preferences of each child are given in Figure 5.2.5. With only one of each snack available, how many different ways can the children each select a snack that they find acceptable? Translating the preferences of Figure 5.2.5 creates the board in Figure 5.2.6.

	cookies	ice cream	peanuts	popcorn	potato chips
Julia	✓	✗	✓	✓	✓
Liz	✓	✓	✓	✗	✗
Mike	✓	✓	✗	✗	✓
Nicole	✗	✓	✓	✓	✓
Scott	✓	✓	✗	✓	✓

Figure 5.2.5

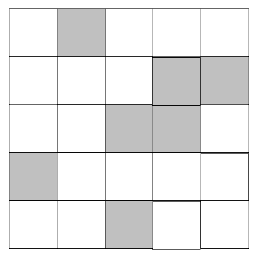
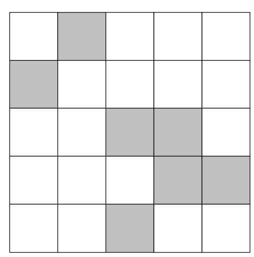


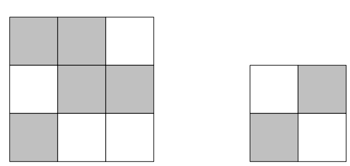
Figure 5.2.6

By switching rows 2 and 4 of Figure 5.2.6 we create the board *B* in Figure 5.2.7 which is then easily decomposed into the two disjoint boards of Figure 5.3.8.



B

Figure 5.2.7



B_1 B_2

Figure 5.2.8

It is not difficult to determine that $R(B_1, x) = 1 + 5x + 6x^2 + x^3$ and simple to find that $R(B_2, x) = 1 + 2x + x^2$. Hence, $R(B, x) = R(B_1, x) * R(B_2, x) = 1 + 7x + 17x^2 + 18x^3 + 8x^4 + x^5$. The number of different legal arrangements is
$\sum_{i=0}^{n}(-1)^i r_i (n-i)! = 5! - 7 * 4! + 17 * 3! - 18 * 2! + 8 * 1! - 1 = 25.$

Homework

1. Find all nine derangements of the first four integers.

2. Using Theorem 5.2.1, compute d_n for $n = 5, 6$.

3. Using Theorem 5.2.2, compute d_n for $n \leq 10$.

4. The manager of eight software engineers wants to perform some quality assurance on the code they have written. If each engineer was the sole creator of one portion of the code, how many ways can their manager assign them to test exactly one piece of someone else's code?

5. Jeff's mother finds 12 cassettes and their shells in his room. Without bothering to match tape to shell, she randomly puts each cassette in a shell. How many ways can she do this and match at least one cassette to its proper shell?

6. A book club with ten members host a swap party where each member brings a different book. All books are deposited into a common box and each member randomly selects one book. What is the probability that no one selects their own book?

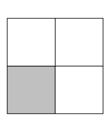
7. Repeat Question 6 for a club with $n = 11, 12, 13, 14, 15$ members. What two things can you say about the probabilities? Hint! Find the reciprocal of each probability.

8. A **fixed point** x of a function f is a point that maps to itself. How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ contain at least one fixed point?

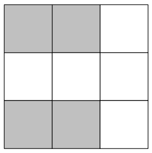
9. Prove $d_n = nd_{n-1} + (-1)^n$ for $n \geq 2$.

10. Find the rook polynomials for each of the following boards.

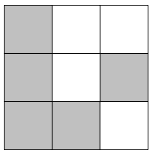
i.



ii.

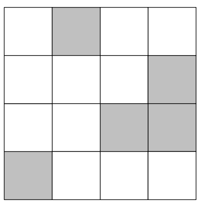


iii.

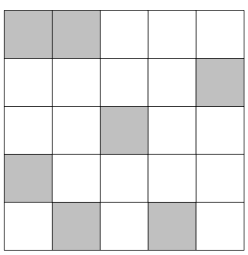


11. Decompose each of the following boards B into two disjoint boards and compute $R(B, x)$.

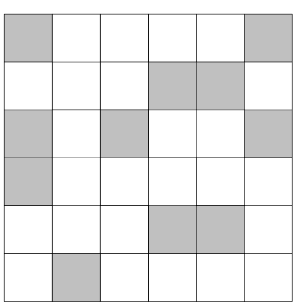
i.



ii.



iii.



12. Tom, the managing supervisor needs to assign six employees to six separate tasks, one employee to one task. The below table indicates that an employee has the skills necessary for the various tasks with a check. How many ways can the Tom assign the tasks to be performed by qualified employees?

	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
Alice	✓	✗	✓	✓	✓	✓
Bob	✗	✓	✓	✗	✓	✓
Mandy	✓	✓	✓	✓	✓	✗
Melissa	✗	✗	✓	✓	✓	✓
Randall	✓	✓	✗	✓	✗	✓
Steve	✓	✓	✗	✓	✓	✓

13. Tourists Amanda, David, Jim, Kim and Scott are visiting New York as part of a group tour. The tour service has a single ticket for five different activities indicated in the table below. Based on the preferences below, how many ways can the group distribute tickets so that each person can enjoy an acceptable event?

	Cats	Giants game	Late Show with Letterman	Rangers game	Yankees game
Amanda	✓	✗	✓	✗	✗
David	✗	✗	✗	✓	✓
Jack	✗	✗	✓	✓	✓
Kim	✗	✓	✓	✓	✓
Scott	✗	✓	✓	✗	✗

14. A small company needs to immediately and simultaneously deliver products to four different locations. This company has four vehicles: a van, a stick shift pickup truck, a sedan and a stick shift sports car. Of the four employees who are available to make deliveries, David can drive any of the vehicles, Joe cannot drive either of the two stick shift vehicles but can drive the other two, Kim can drive any type of vehicle but refuses to drive the truck because it has no air conditioning and the van because it has no radio and Sarah will only drive a stick shift vehicle. How many ways can employees be matched to the vehicles so the deliveries can be made simultaneously?

15. Five coin collectors win an online auction of five coins, a 1900 silver Morgan dollar, an uncirculated silver 1964 Kennedy half dollar, a 1955 silver Franklin half dollar, an uncirculated 1943 zinc Lincoln penny and a 1905 Indian head penny. Mike only collects uncirculated coins, Donna only collects pre-1950 coins, Jennifer only collects half dollars, Calvin collects only pennies and Joe only collects silver coins. How many ways can they take exactly one coin for their collections?

16. The Jones family has five chores to be finished today, clean the garage, vacuum the rugs, grocery shopping, mow the yard and pay the bills. Only the parents are authorized to

write checks for the bills. Only the parents and Emily have a driver's license and can go grocery shopping. If Emily goes grocery shopping she will pay with cash and is also content to vacuum. But, Emily complains too much when she mows the yard or cleans the garage, so everyone is happier if she doesn't perform those chores. John and Chris refuse to vacuum but are willing to clean the garage or mow the yard. The parents, of course, can and will do any chore necessary. How many ways can each family member be assigned exactly one chore?