4.1 Chromatic Number

There are three types of problems in combinatorics: the existence problem, the counting problem and the optimization problem. Essentially the three are distinguished by the questions asked. First, does there exist an arrangement (object) that has certain characteristics. Second, how many arrangements (objects) have certain characteristics. Third, among all possible arrangements (objects) with certain characteristics, which is the best based on some criterion. Until now this book has been concerned with the first two questions, this chapter focuses on the third question.

As with other graph theory topics, the concept of coloring graphs began with a puzzle called the Four-Color Problem. Succinctly, it states that coloring a map such that no two neighboring countries are colored the same color requires at most four colors. The problem hit the mathematical radar when Augustus DeMorgan wrote a letter to William Rowan Hamilton on October 23, 1852. The search for a proof of this result has fueled advances in graph theory ever since. It has also involved controversy because the 1976 proof by Wolfgang Haken and Kenneth Appel used over a thousand hours of computer time. *Four Colors Suffice* by Robin Wilson traces the history of the problem. One of the fertile areas developed because of this problem is colorings of graphs.

Consider the following scenario. Seven coworkers are organizing a lunch excursion. Nonsmokers Ben, Rick and Susan refuse to ride with smokers Amanda, John and Kim. Jill is ambivalent about smoking but won't ride with Susan or Kim because she is allergic to their perfume. Ben and Susan once dated and won't ride in the same car. What is the minimum number of cars needed for this lunch excursion?

This problem can easily be modeled with a graph (see Figure 4.1.1). Let the vertices of the graph represent the seven coworkers. Two vertices will be adjacent if the two represented individuals will not share a ride.

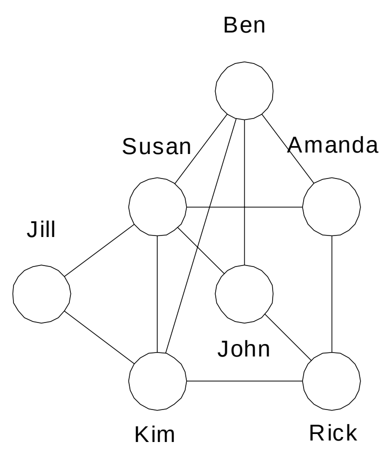


Figure 4.1.1

The assignment of individuals to different cars with the given restrictions requires a partition of the vertices into different sets. Since adjacent vertices mean the two coworkers will not ride together a partition must place adjacent vertices in different sets. Table 4.1.1 indicates one possible partition using three sets, or in other words three cars.

Car	Occupants
1	Ben, Jill, Rick
2	Susan
3	Amanda, John, Kim

Table 4.1.1

In fact, there may be many different acceptable partitions. Rick could easily be assigned to car two for another possible solution. Again, three cars are needed. Is three the minimum number of cars needed? In this case, yes. Clearly, Jill, Susan and Kim cannot ride with one another because their vertices form a K_3 subgraph forcing the use of at least three cars. The solution uses three cars which means three is the minimum.

The example given is small and the partitions are easily identified. For dealing with larger graphs it would help to organize the information visually. Suppose the cars are represented by colors. Let car one be red, car two be blue and car three be green. Then the solution in Table 4.1.1 can be represented by the colored graph in Figure 4.1.2 where r, b and g represent the red, blue and green respectively. These thoughts give rise to the following definition. For any graph $G = (V, E)$, a **coloring** of G is an assignment of colors to each vertex of G.

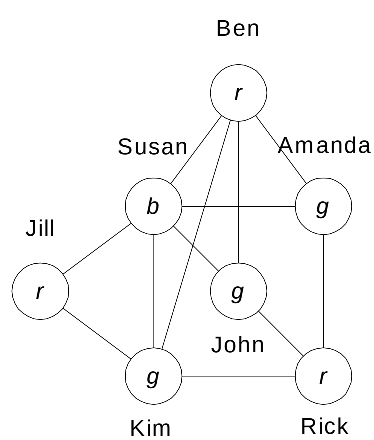


Figure 4.1.2

Figure 4.1.3 shows two colorings of the same graph. Now, in the example, adjacent vertices represent people who cannot be in the same car. In other words, adjacent vertices cannot be colored the same color. To formalize this, a **legal coloring** of G is an assignment of colors to each vertex of G such that adjacent vertices are assigned different colors. The legal coloring in Figure 4.1.3 is labeled.

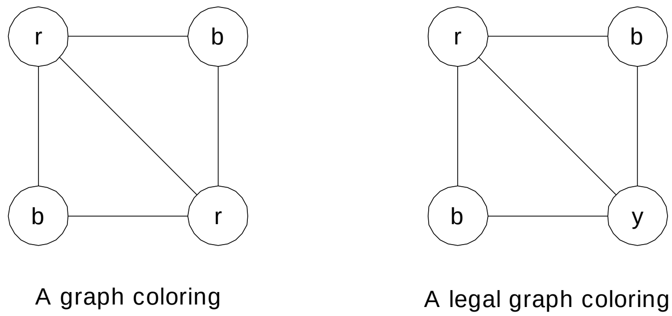


Figure 4.1.3

Returning to the car example, everyone could drive in separate cars resulting in seven colors, one for each vertex. This is called a 7-coloring of the graph in Figure 4.1.1. A ***k*-coloring** of a graph G is a legal coloring using at most k different colors. A k-coloring is used to define the minimum coloring for a graph. The **chromatic number of a graph *G*,** $\chi(G)$, is the integer k such that G can be k-colored but cannot be $k-1$ colored. Throughout the rest of the text the term graph coloring will mean a legal graph coloring. Let's use $K_{2,3}$ to illustrate the differences between a k-coloring of G and $\chi(G)$.

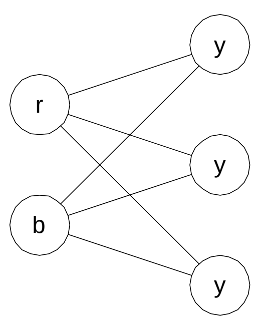


Figure 4.1.4

In Figure 4.1.4, $K_{2,3}$is shown with a 3-coloring which says $\chi(K_{2,3}) \leq 3$. Is it possible to color it with fewer colors? Changing the blue colored vertex to red results in the 2-coloring in Figure 4.1.5 and $\chi(K_{2,3}) \leq 2$.

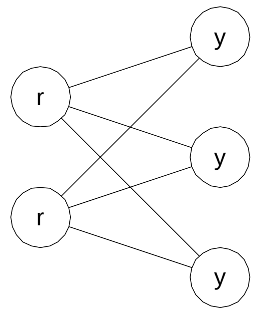


Figure 4.1.5

Can this be improved to a 1-coloring? No. The only graphs that can be 1-colored are the null graphs because they contain no edges. Since $K_{2,3}$ is not a null graph, $\chi(K_{2,3}) \geq 2$. Together, these two inequalities force $\chi(K_{2,3}) = 2$. The example demonstrates the way to prove $\chi(G) = k$. First, a k-coloring must be exhibited to show $\chi(G) \leq k$. Second, an argument is required to show that there is no $k-1$ coloring giving $\chi(G) \geq k$. The only way to satisfy both inequalities is for $\chi(G) = k$.

In general it is difficult to find the chromatic number of a graph. In fact, determining $\chi(G)$ is in the NP-complete class of problems. The remainder of the section focuses on the chromatic number of common classes of graphs such as the complete graphs, cycles, and wheels. It also provides some useful strategies that sometimes result in the chromatic number of a graph. Section 4.2 introduces chromatic polynomials where evaluating the polynomial counts the number of k-colorings and Section 4.3 provides several polynomial time algorithms that correctly calculate $\chi(G)$ for some graphs but not all.

As stated earlier, $\chi(N_n) = 1$. On the opposite end of the spectrum, $\chi(K_n) = n$. Why is this? Naturally, every graph with n vertices can be legally n-colored to show $\chi(K_n) \leq n$. A coloring of K_n with $n-1$ colors would force two vertices to receive the same color since any two vertices are adjacent in K_n. Therefore, no legal $n-1$ coloring exists and $\chi(K_n) \geq n$.

Consider C_5 and C_6 in Figure 4.1.6. Begin coloring at the top vertex with the color red and continue coloring the cycle clockwise. A new color will be used only if it is required. For C_6 the first vertex is red, the second is blue, the third is red, the fourth is blue, the fifth is red and the sixth is blue. Therefore $\chi(C_6) \leq 2$ and the fact that it is not a null graph gives equality. Next, color C_5. Proceeding in the same manner, the first vertex is colored red, the second is colored blue and the third is colored red. However, the fifth vertex is adjacent to both a blue and a red vertex and must be colored a third color, namely yellow. As can be seen, the chromatic number of C_n is determined by the parity of n and $\chi(C_n) = \left\{ \begin{array}{ll} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{array} \right\}$.

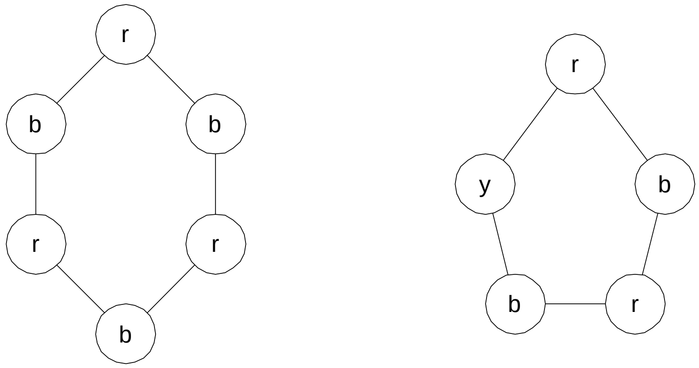


Figure 4.1.6

The graph in Figure 4.1.1 has K_3 as its largest complete subgraph and the chromatic number is three. The complete graph's largest complete subgraph is itself and $\chi(K_n) = n$. The null graphs have largest complete subgraph $K_1$and chromatic number one. The bipartite graph studied above has largest complete subgraph K_2 and $\chi(K_{2,3}) = 2$. It seems like a pattern is developing. But C_5 has K_2 as its largest complete subgraph and its chromatic number is three. While the pattern does not hold, an important tool in determining the chromatic number of a graph is to look for the largest complete subgraph and cycles of odd length. If K_n is the largest complete subgraph for G, then $\chi(G) \geq n$. If G contains a cycle of odd length, then $\chi(G) \geq 3$. Let's illustrate with several examples.

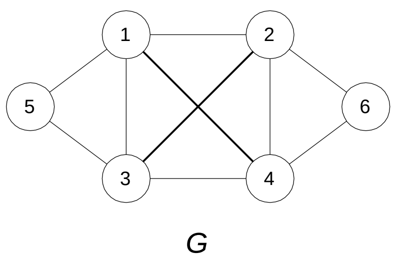


Figure 4.1.7

In the graph G of Figure 4.1.7 the vertices 1, 2, 3 and 4 form a K_4. Thus, $\chi(G) \geq 4$. The coloring of the graph begins by coloring the K_4 subgraph with four different colors and then coloring the other two vertices. The 4-coloring of G in Figure 4.1.8 shows that $\chi(G) \leq 4$ and these inequalities prove $\chi(G) = 4$.

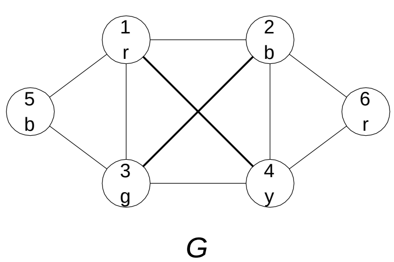


Figure 4.1.8

Now, consider the graph in Figure 4.1.9. The largest complete subgraph of this graph is K_3. So $\chi(G) \geq 3$. Unlike the previous example, the chromatic number does not equal the size of the largest complete subgraph. Instead, the cycle of length five holds the key to the solution. The coloring of G begins with the vertices 1, 2, 3, 4 and 5. They form a C_5 which requires three colors. Vertex 6 is adjacent to each of those 5 vertices and forces the use of a fourth color. Hence, $\chi(G) = 4$ even though the graph G contains numerous K_3's and not a single K_4. The graph G with $n+1$ vertices is called a wheel with n spokes, denoted by W_n, because a single central vertex, called a hub, is adjacent to the other n vertices in the graph which form a cycle called a rim. The graph in Figure 4.1.9 is the wheel W_5.

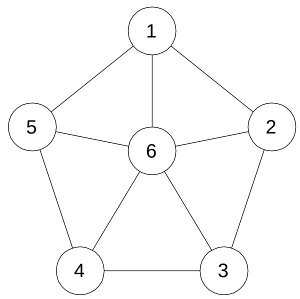


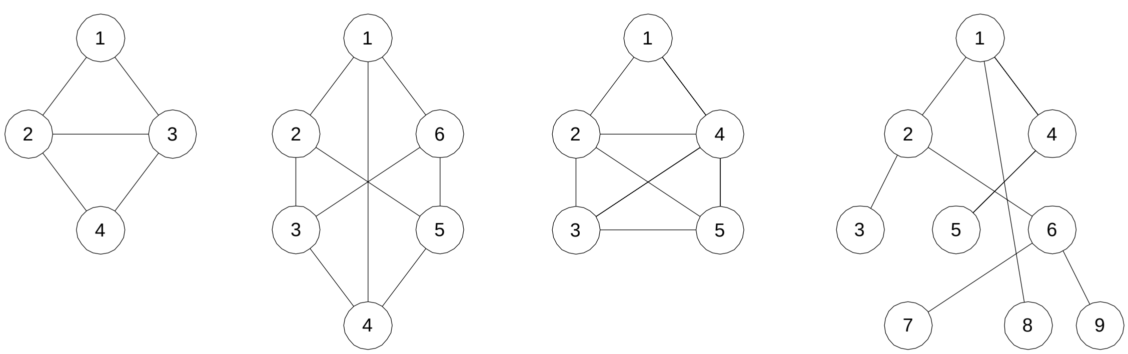
Figure 4.1.9

As can be seen, there are some strategies for finding the chromatic number of a graph but they do not always work. Section 4.3 returns to this topic and introduces some efficient algorithms that sometimes find, and other times only get close to, the chromatic number. But a question thus far not asked is: given a graph, how many k-colorings of the graph exist? Section 4.2 partially answers this question.

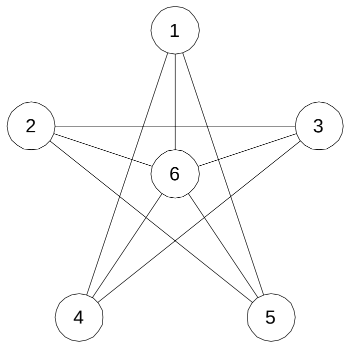
Homework

1. Find the chromatic number for each of the graphs below.

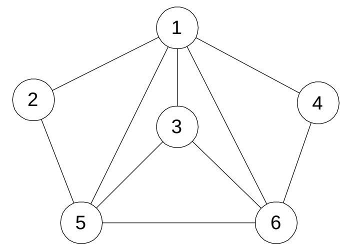
A *B* *C* *D*



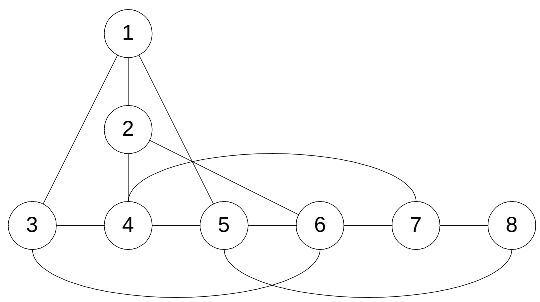
E



F

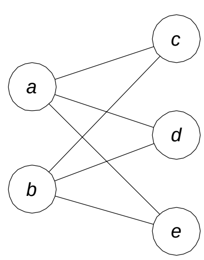


G

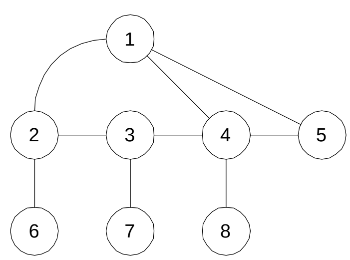


2. Find the chromatic number for each of the graphs below.

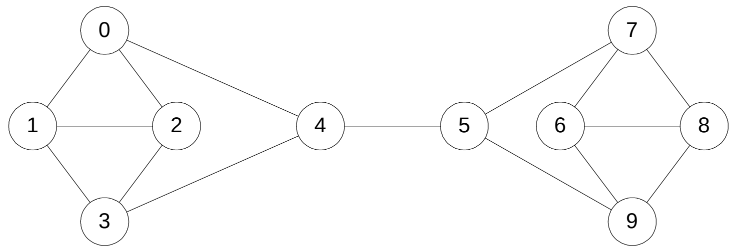
A



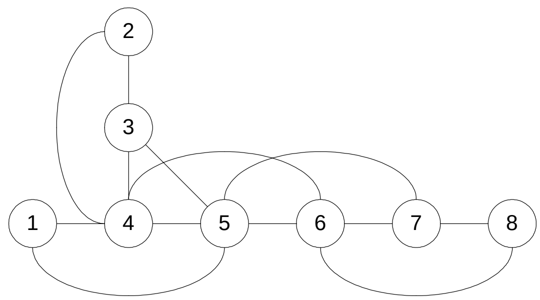
B



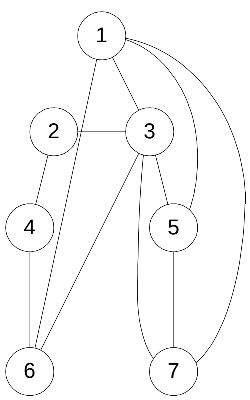
C



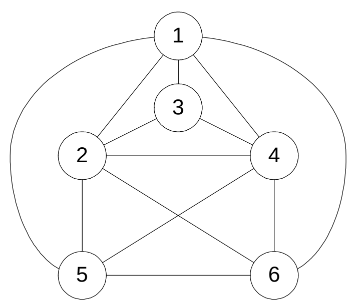
D



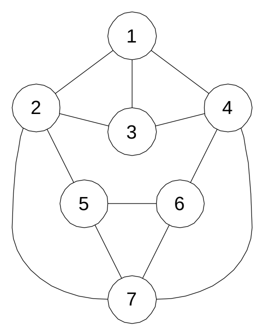
E



F



G



3. What is the minimum number of cars needed for the lunch excursion in the first example of this section if John and Amanda had once dated and refused to ride with each other?

4. Seven faculty members each serve on some subset of six committees as outlined in the table below. Naturally, no faculty member can be in more than one place at a time. What is the minimum number of meeting times needed for these committees?

	1	2	3	4	5	6
Bell	✓	✓				
Burke	✓	✓			✓	
Ellermeyer					✓	
Laval	✓		✓	✓		
Morgan		✓	✓			
VanBrackle	✓			✓		✓
Zumoff		✓				✓

5. At the law firm of Brackman, Dylan, Fisk, Murdock and Young, six clients need to have day long meetings with their attorneys. Trask Industries is represented by Fisk and Murdock, Crais International is represented by Brackman and Dylan, Spam.Com is represented by Fisk and Young, E. Cole Enterprises is represented by Brackman and Murdock, The S. Plum Security Firm is represented by Dylan, Murdock and Young while Tinker Steel Manufacturers is represented by Dylan and Young. If no lawyer can attend two different meetings in a single day, what is the fewest number of days needed to schedule these meetings?

6. Andy is going to set up some aquarium tanks at home for six species of tropical fish. Naturally, Andy does not wish to house any species that would prey on each other in the same tank. Suppose species 1 feeds on species 2 and 5; species 2 feeds on species 1 and 5; species 3 feeds on species 4 and 5; species 4 feeds on species 2; and species 5 and 6 do not feed on any other species. What is the minimum number of tanks Andy will need?

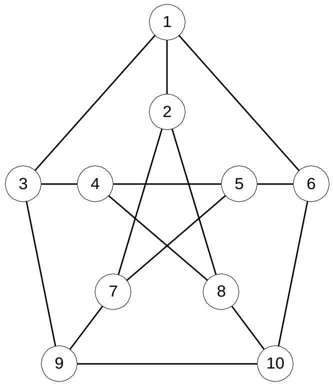
7. Determine $\chi(K_{n,m})$.

8. Determine $\chi\left(\overline{K_{n,m}}\right)$.

9. Prove $\chi\left(\overline{C_n}\right) \geq \lfloor \frac{n}{2} \rfloor$ for $n \geq 3$.

10. Determine $\chi(W_n)$.

11. What is the chromatic number of the Petersen Graph?



12. What is the chromatic number of a tree?

13. Randomly select a graph G from the collection of all non-isomorphic graphs with four vertices. Compute the probability that $\chi(G) = 2$.

14. Let G be a graph. Prove $\chi(G) \geq \omega(G)$. Construct a graph G where $\chi(G) > \omega(G) \geq 3$.

15. Let G be a graph. Prove $\chi(G) \leq n - \alpha(G) + 1$. Construct a graph G where $\chi(G) < n - \alpha(G) + 1$.