Math 1106 – Elementary Applied Calculus
Textbook: Bittinger's Calculus and Its Applications
Lecture Notes and Suggested Exercises for Chapter 4, "Integration"

Added after Day 22 class meeting, Monday, November 2, 2009

In the preceding chapters, we've studied "Differential Calculus": that is, we've been starting with a function named *"f"* and deriving an associated function that's been named with the same letter but with a prime attached to the name to distinguish it from the original function. This derived function, called the derivative, is simply able to report on the first function's "slope" values. Differential Calculus is concerned with slope: slope, slope, slope, slope, slope, slope, slope – that's what all those "rules for differentiation" enable us to do: get the "slope function."

Now, in Chapter 4, we begin to investigate that other side of calculus that's called "Integral Calculus." In the days ahead, it will become clearly evident that the two types of calculus are two sides of the same coin; they are the yin and yang of calculus; each is the peanut butter to the other's jelly.

As I explained in class, "Differential Calculus" is what Isaac Newton developed to mathematically prove what Kepler had said about planetary motion — that heavenly bodies revolve around others in orbits that are not perfectly round, but are rather elliptical in shape; and that in the course of the orbit, as the smaller body orbits the larger body it will speed up and then slow back down after it has passed by the larger body. Newton, using that new-fangled gadget, the telescope, was able to take careful measurement of planetary positions night after night, and from all that data he was able to deduce that the planetary orbits are indeed slightly elliptical, and that the speeding up phenomenon actually happens. Remember: for a function that measures position as a function of time, the first derivative measures speed as a function of time. That's why he invented Differential Calculus — to exploit the derivative in order to measure the speed of the orbiting planet, and show that it did indeed vary in the way that the astronomer Kepler said it would.

Now, in this chapter, we'll see that Integral Calculus deals with calculating the area of various shapes — specifically, it will measure the area between a function's graph and the x-axis, between two specific values of x. In class, I showed how Kepler had made another assertion about figures "swept out" by the orbiting satellite, and how they would have the identical area.

For some types of figures, calculating the area is as simple as multiplying width times height (for rectangles); triangular shapes can be handled just as easily if we remember that the area of a triangle is one-half the product of the base times the height of the triangle. The area of a trapezoid may not be quite as familiar to most people, but it's equal to the product of the base times the average of the heights of the two sides. But when the function has a graph that is something other than a straight line, we leave the comfortable realm of rectangles and triangles and trapezoids and we have to be clever to deduce that enclosed area. Clever — like Newton was.

I gave a glimpse into why Newton was curious about measuring the area of odd-shaped figures as a result of his trying to further prove the assertions of the earlier astronomer named Kepler. We will find that measuring areas has many other applications (in business!) than simply astronomical questions.

The technique Newton pioneered was to simply overlay a series of equal-width rectangles over the desired figure, with height of the rectangle being governed by the height of the relevant function for the value of x that corresponds to the left-hand edge of the rectangle. Summing up the individual areas of the overlaid rectangles will give a number that is not too far off the actual area of the region being investigated — and increasing the number of rectangles (by choosing a smaller common width for them) leads to better and better approximations of the area of the region being investigated.

Look at Exercise 23 on page 399 of the textbook, and concentrate first on part (a). That asks you to approximate the area between the x-axis and the graph of $f(x)=\dfrac{1}{x^2}$ between the x-values of 1 and 7. That means you are to calculate $\sum_{i=1}^{n} f(x_i)\,\Delta x$ with n = 6, and in this case, that makes $\Delta x=\frac{7-1}{n}=\frac{6}{6}=1$. Well, that means all the rectangles have a common width (Δx) of 1, and the summed area of all 6 rectangles will be:

$$\sum_{i=1}^{6} f(x_i)\cdot\Delta x = \left(\tfrac{1}{1^2}\cdot 1\right)+\left(\tfrac{1}{2^2}\cdot 1\right)+\left(\tfrac{1}{3^2}\cdot 1\right)+\left(\tfrac{1}{4^2}\cdot 1\right)+\left(\tfrac{1}{5^2}\cdot 1\right)+\left(\tfrac{1}{6^2}\cdot 1\right)\cong 1.4914$$

(for goodness sake, use your calculator!).

But now, in part (b), you're told to get a better approximation of that area by using 12 rectangles instead (that means that $\Delta x=\frac{7-1}{n}=\frac{6}{12}=\frac{1}{2}$). The calculation thus becomes

$$\sum_{i=1}^{12} f(x_i)\cdot\Delta x=\sum_{i=}^{12}\left(\tfrac{1}{x^2}\right)\cdot\tfrac{1}{2}=$$

$$\left(\tfrac{1}{1^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{1.5^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{2^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{2.5^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{3^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{3.5^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{4^2}\cdot\tfrac{1}{2}\right)+$$
$$\left(\tfrac{1}{4.5^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{5^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{5.5^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{6^2}\cdot\tfrac{1}{2}\right)+\left(\tfrac{1}{6.5^2}\cdot\tfrac{1}{2}\right)$$
$$=\left(\tfrac{1}{1^2}+\tfrac{1}{1.5^2}\cdot\tfrac{1}{2^2}\cdot\tfrac{1}{2.5^2}+\tfrac{1}{3^2}+\tfrac{1}{3.5^2}+\tfrac{1}{4^2}+\tfrac{1}{4.5^2}+\tfrac{1}{5^2}\cdot\tfrac{1}{5.5^2}\cdot\tfrac{1}{6^2}\cdot\tfrac{1}{6.5^2}\right)\cdot\tfrac{1}{2}\quad\text{[since you can factor out the }1/2\text{!]}$$
$$\cong 1.1418$$

(you'd be crazy not to use your calculator for that!).

You'll be asked to do one like this in your MyMathLab homework. You'd be crazy not to use your calculator!

And there are features of your TI-83 which will reduce the tedium quite a bit. I'll describe a procedure here for doing such a summation, in case you're willing to be curious and try them. (And when the number of rectangles being used gets even just a little bit big, you'll be thankful you've got that calculator and know these "tricks"!)

Here are the steps:

1. Put the function involved into Y_1.

2. Store the value of the left-most rectangle's left edge into the storage area of your TI-83 called A, using the STO button (that's the black button just above the ON button on the left) and the ALPHA key followed by the MATH button.

3. Store the value of the right-most rectangle's right edge into the storage area of your TI-83 called B, using the STO button.

4. Determine the width of the rectangles to be used, either because you're told what it is, or because you're told how many of them to use. (In the latter case, simply divide the difference between x=b and x=a by the number of rectangles.) Store that result into the storage area of your TI-83 called D, using the STO button.

5. Tell the TI-83 to do all that arithmetic for you:

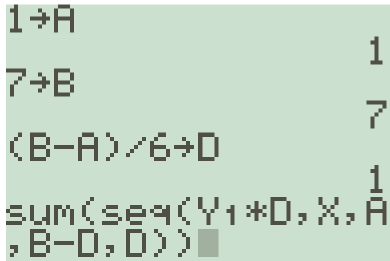
$$\text{sum(seq(}Y_1\text{*D,X,A,B-D,D)).}$$

(Pay attention to that next-to-last parameter. It is B-D, since we want the final rectangle's left-hand edge to be used to evaluate the function.)

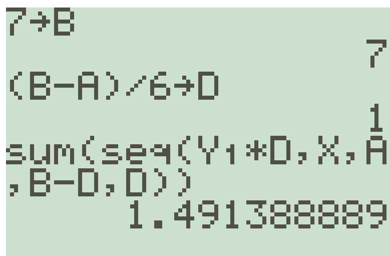
Then, if you want to reduce the value of D (that is, to use more rectangles that are skinnier), store the new value into D and merely re-execute that final command for the new result. That will cut down on the amount of grungy arithmetic.

Where are the commands "sum" and "seq" found in your TI-83? Watch carefully. While you're on the work screen, press 2^{nd} STAT (which is that yellow LIST alternate function for the STAT button) and cursor right twice to the MATH menu, and choose option 5, SUM. Then immediately press 2^{nd} STAT again, but this time only cursor once to the right to bring up the OPS menu. Choose option 5 here, too – it's the SEQ function. Then fill in the 5 parameters in the correct order, separated by a comma between each parameter.

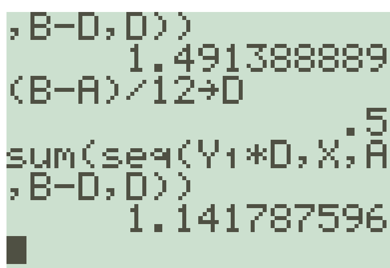
Here's the screen of my TI-83 (after having stored the function into Y_1) as I prepared to do part (a) of Exercise 23 on page 399 and just before pressing the enter key:



Then I pressed the Enter key and this was the answer:



After I redid the calculation of D for 12 rectangles, I pressed the yellow 2nd button and then the ENTER key, and pressed them in that order one more time, which brought back that mysterious line involving SUM and SEQ. But because I had, just prior to that, stored a new value for D, it calculated the improved answer:



Now go for the gold! Change the value of D to 64 – that's 64 little rectangles all snuggling next to the graph of the function. The TI-83 calculated their combined area to be about 0.9045, which is getting fairly close to the actual area figure of about 0.8571 (and in a later section, you'll learn how you, too, can precisely measure the area using the features of Integral Calculus that you'll be learning soon).

See you Wednesday — do homework before hand, and then bring questions to class!!!

Added after Day 23 class meeting, Wednesday, November 4, 2009

Today I went over some of the material in Section 4.2, showing how Newton may have progressed to understanding that any formula for the area of the region between a function's graph and the x-axis, expressed as a function, has a derivative that is the super-family of all functions that include the original function — meaning that they are all simply the original function with the addition of a constant term.

Added after Day 24 class meeting, Monday, November 9, 2009

Because it looks like the antiderivative is what Newton discovered was needed to measure the relevant area, Section 4.2 also introduces some simple rules for antiderivatives (see page 406).

The symbolic notation that tells us we're supposed to get the "antiderivative" of a function of *x* (that is, *f(x)*) looks like this: $\int f(x)dx$.

There won't be just one antiderivative, since the derivative of a constant function is always zero. So, if the function *F(x)* is <u>some</u> antiderivative of the function *f(x)* (that is, *f(x) = F'(x)*), then we will write $\int f(x)dx = F(x) + C$ to indicate the whole infinitely-large family of antiderivatives of the function *f(x)*. The capital "C" is called the "constant of integration." The process of getting the antiderivative of a function is called "integration" and we're said to be "integrating the function."

The most ubiquitous rule of integration (Rule #2 on page 366) is the reverse-power rule. If *r* is any number ***<u>except</u>*** -1, then the rule says this:

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C.$$

The other 3 important rules in the upper blue box on page 366 are easily seen to be reversals of the easier rules of differentiation.

Once you've gotten the antiderivative, you can check your answer by taking the derivative of that and seeing if you get back to the original function you started with!

Added after Day 25 class meeting, Wednesday, November 11, 2009

Today was the review day for Test #3

Added after Day 26 class meeting, Monday, November 16, 2009

Today was the day you took Test #3

Added after Day 27 class meeting, Wednesday, November 18, 2009

Gosh, could it be?!? Can the antiderivative of a function really be used to easily calculate the area between the graph of the function and the *x*-axis over a stretch of the *x*-axis from *a* to *b*? That is some wild and weird coincidence, don't you think?

Summary: If $F'(x) = f(x)$ (which means *F* is *any* anti-derivative of *f*), then the **definite integral,** $\int_a^b f(x)dx = F(b) - F(a)$, is the easy calculation for the area.

Notice that you can tell when a <u>definite</u> integral (that is, a number) is being called for because the integral-symbol has those limits of integration appended to its top and bottom. Make sure you get the order of the subtraction correct: First, use the top limit (b) and then subtract the value associated with the lower limit (a).

Also, there's a bit of explaining to be done concerning what it means if the graph of $f(x)$ dips down below the *x*-axis anywhere between *x*=*a* and *x*=*b*. It seems that the concept of "area" needs to encompass the notion of "negative" area. Any region that lies below the x-axis is calculated with a negative value!

Two important applications of the definite integral are introduced in Section 4.4. One of them is the extension of the measurement of "area" to the idea of calculating that area that lies between the graphs of two functions. The application shown on page 430 indicates how the measurement of a definite integral (the "area") is a way to measure the net savings in pollution particulates over a period of time when using a better type of internal combustion engine. The other idea is the calculation of the average value ("height") of a function thru the use of the definite integral divided by the span over which the integral is calculated.

For next time, be sure to read the "Technology Connection" on page 418. It holds the key to why I will not be covering any of the material in Sections 4.5-4.7.