Math 1106 – Elementary Applied Calculus
Textbook: Bittinger's *Calculus and Its Applications*
Post-Lecture Notes for Chapter 2, "Applications of Differentiation"

A critical observation about Chapter 2 as a whole:

This chapter is 106 pages in length. It screams to have a "Cliff's Notes" version constructed. The authors of the textbook are garrulous, prolix, suffers from logorrhea, and is also verbose; not only that, they use a lot of words. ;-) I'll try to express the important things about this chapter in a minimum of words below.

This chapter teaches techniques for determining when a given function either reaches its absolute lowest or absolute highest between two given values of *x* — locating its "peaks" and "valleys." Where we're going is NOT really about learning how to sketch the graph of the function (we can let our calculator do that). But rather, with some observations about the derivative (and the second derivative) of the given function, we can quickly determine the coordinates of peaks and valleys for the function.

Added after Day 13 class meeting, Wednesday, September 30, 2009

Summary of Section 2.1: The peaks and valleys on the graph of $y = f(x)$ will only occur where $f'(x) = 0$ or $f'(x)$ does not exist. Note: sometimes even those are not peaks or valleys, but at least those are the only places you need to check for peaks and valleys. Those places are called *critical points,* and the values of the input variable for them are called the *critical values* of the function. **End of Summary.**

A "critical value" of a function is a value of "x" where either the derivative of the function does not exist, or the derivative has a value of zero. Between successive critical values, the derivative will either always be negative, or else it will always be positive; one or the other, exclusively.

The places where a function's graph reaches either a high spot or a low spot (relative to the near-by surrounding points of the graph) occur precisely where critical values occur. (Not every critical value corresponds to a relative maximum or relative minimum, but at least those relative extrema only occur for critical values, so the hunt is contained.)

Between successive critical values on a graph, the derivative always exists and it is either always positive or always negative.

The multi-color graph on page 206 shows a very important result for the given function. Notice that the red function is a cubic polynomial, and the blue function is its derivative. Notice that for every spot on the red curve corresponding to a critical value, the blue curve is crossing the x-axis: that shows pictorially that the relative extrema of the red curve occur only where the critical values of the red curve are.

The Technology Connection on page 210 gives a practical way of pinpointing the relative extrema for a curve. Yes, not only can your calculator graph the function, it can do a darn good job of pinpointing the coordinates for the peaks and valleys.

The graph of the function shown on page 209 is an example of a function that has a critical point that is a place where the derivative does not exist! Notice that the curve is continuous at the point (2,1) but it is not SMOOTH. If a function is not SMOOTH at a particular spot (it bends suddenly), then the derivative is not defined for that value of *x*.

Added after Day 14 class meeting, Monday, October 5, 2009

Section 2.2 brings the second derivative into the picture, to help in determining whether a critical point represents a peak or a valley.

Summary of Section 2.2: Suppose "c" is one of those critical points for the function $y = f(x)$ where $f'(x) = 0$; that is, $f'(c) = 0$. The "second derivative test" says that when $f''(c) > 0$ you've found a valley; if instead $f''(c) < 0$ it means you've found a peak. But if $f''(c) = 0$ then no conclusion can be drawn about whether the graph is a peak or a valley. **End of Summary.**

One more important word is in order to supplement the summary: that word is "concavity." For the stretches where $f''(x) > 0$, the curve is said to be "concave up". Where ever $f''(x) < 0$ the curve is said to be "concave down." The only places where concavity can change (from being "concave up" to being "concave down" or *vice versa*) is where either $f''(x) = 0$ or where the second derivative does not exist. A spot on the graph where concavity changes is called an "inflection point."

Added after Day 15 class meeting, Wednesday, October 5, 2009

Section 2.3 is about rational functions, and they do provide a additional rich source of functions we will be investigating for their extrema. Today, I gave a Cliff's Notes version of the information in this section, as well as Section 2.4. I also noted that Section 2.6 will be shifted to the next block of instruction after the coming test, so you'll see that the due date for that homework has been extended out into November. Section 2.5 is the crescendo for this chapter, and it is critically important that you work on that homework even though I won't talk about the section until next Monday. You already have the tools to work on those word exercises: find the peaks and valleys!

I promised to post the "Cliff's Notes" that I created for investigating asymptotes, so you'll find those below:

Suppose $f(x) = \frac{P(x)}{Q(x)}$, where both $P(x)$ and $Q(x)$ are polynomials.

Suppose further that the leading coefficients of $P(x)$ and $Q(x)$ are a and b respectively.

Finally, refer to the degrees of these two polynomials as deg(P) and deg(Q). Reminder: the degree of a polynomial is simply the highest power of its input variable.

1. The function f(x) will have a **vertical** asymptote for any value of x for which Q(x) = 0, provided that P(x) is not equal to zero for that value of x.

2. The function f(x) will have a **horizontal** asymptote provided that deg(P) ≤ deg(Q).

    a. If deg(P) < deg(Q) then the x-axis is the horizontal asymptote.

    b. If deg(P) = deg(Q) then the line $y = \frac{a}{b}$ is the horizontal asymptote.

3. A **slant** (or **oblique**) asymptote occurs if deg(P) = 1 + deg(Q). The expression for the slant asymptote's equation is the quotient obtained by dividing P(x) by Q(x).

So in class, I worked through several of the homework items for Section 2.3 — and after the first few, they turned into multi-part questions. When extrema were sought, that's when the calculus learned earlier in the chapter came into play — I exploited the First Derivative test (to answer questions about increasing and decreasing intervals for the given function) and the Second Derivative (to find inflection points and answer questions about concavity over intervals for the given function).

The important aspect of Section 2.4 was to note that when the domain of a function is restricted, the end-points of the given interval must be considered as *critical values* when looking for the absolute maximum or the absolute minimum of a function. Remember: critical values are those values in the domain of a function where relative extrema may occur.

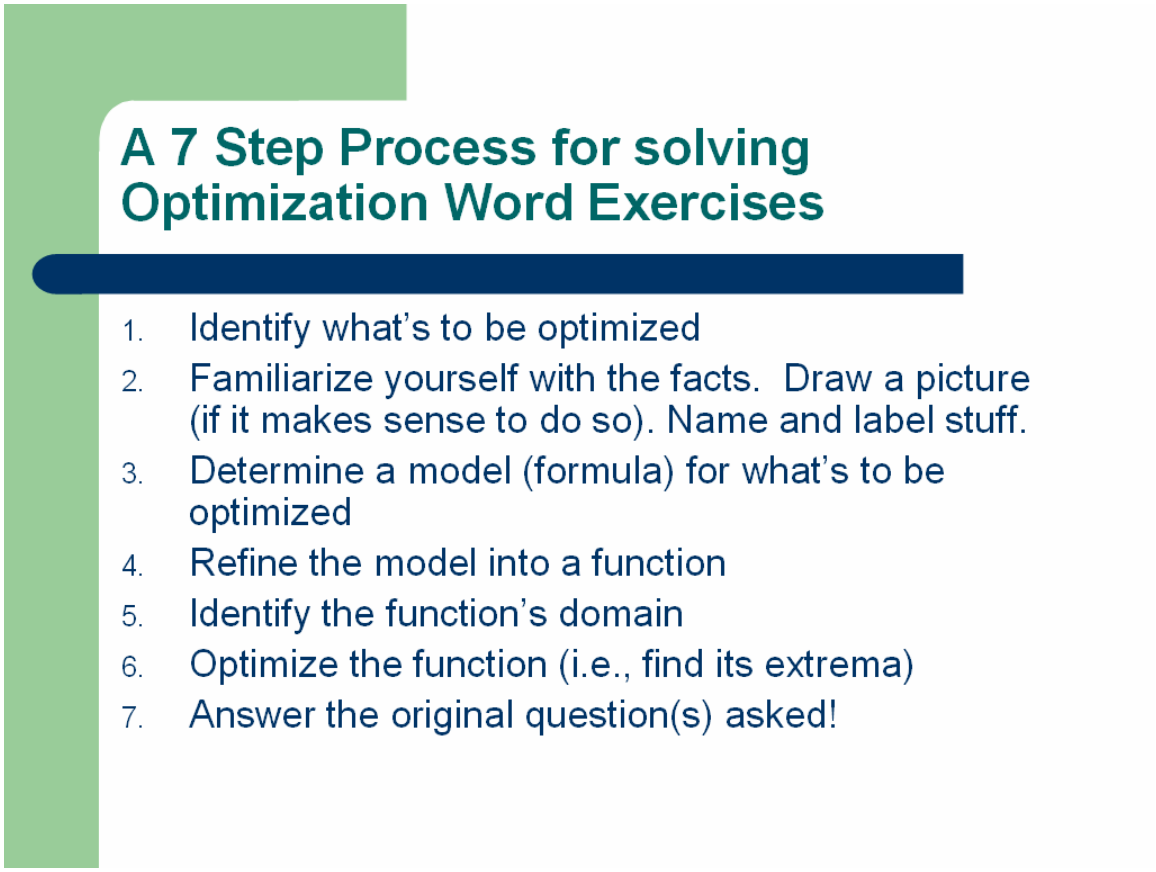
I demonstrated a lot of things having to do with the TI graphing calculator, and I do expect my students to become increasingly proficient in using that as a tool.

See you next Monday. Do your homework! Prepare questions for those homework exercises in Section 2.5, and I'll work on getting a practice test out onto my website by sometime this coming weekend.

Added after Day 16 class meeting, Monday, October 12, 2009

Section 2.5 is the most important section of the chapter. It is full of practical real-world situations about finding where a function peaks or has a valley (a maximum or minimum). For business students, this should be a real important challenge … looking for what value of the input causes the profit function to be at a maximum (for example), or a cost function to reach a low point.

I shared with the class a PowerPoint slide that gives my own 7-step process for successfully navigating and working out the answer to this type of question ("Max-Min"). Here it is:



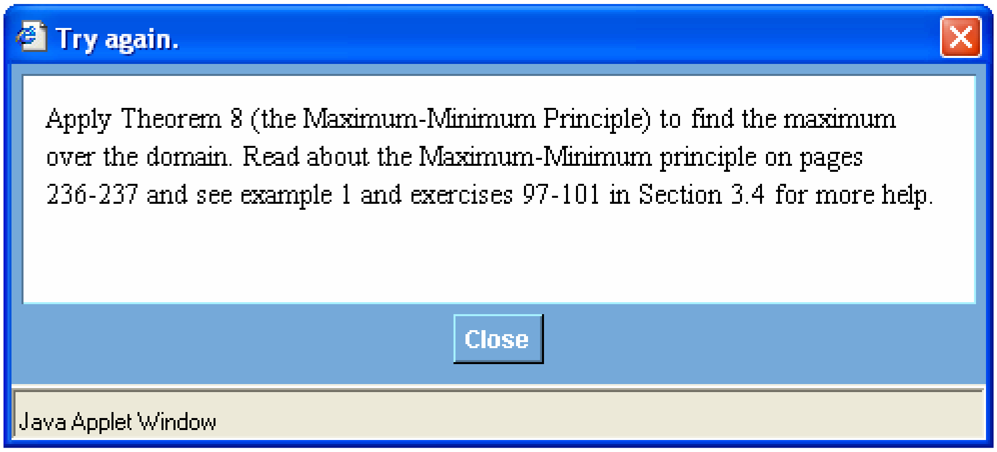
A 7 Step Process for solving Optimization Word Exercises

1. Identify what's to be optimized
2. Familiarize yourself with the facts. Draw a picture (if it makes sense to do so). Name and label stuff.
3. Determine a model (formula) for what's to be optimized
4. Refine the model into a function
5. Identify the function's domain
6. Optimize the function (i.e., find its extrema)
7. Answer the original question(s) asked!

Once you've gotten successfully to step 6, you're home free! The challenge is the formation of the relevant function to be "optimized." (Be sure to read the section on Inventory Cost, because that shows the way to approach finding out how to minimize inventory cost in the case where a store spreads the delivery of a year's worth of product smoothly over the year with reorders, thereby striking the correct balance between paying reorder costs vs. paying on-site storage costs).

Compare my 7-step process to the 4-step process that the authors of our textbook lay out at the bottom of page 260.

The last step is really an important one. In a previous semester (using a different textbook and different homework software), I 'fessed up to having forgotten this step when I wrote:

> "A wonderfully relevant example of the importance of that last step of the process was encountered in the last of the classes I met with yesterday, when I went through one of the homework exercises. The exercise was #16 for Section 3.4. Here are the instructions for that exercise:
>
> > An employee's monthly productivity M, in number of units produced, is found to be a function of the number t of years of service. For a certain product, a productivity function is given by
> >
> > $$M(t) = -2t^2 + 120t + 170, \quad 0 \le t \le 40.$$
> >
> > Find the maximum productivity.
>
> "I was so focused on the restricted domain aspect of the exercise, that when I figured out the only critical point to be *t = 30*, I keyed in "30" as the answer and was shaken to get back the computer equivalent of a sneer.
>
> 
>
>
> Apply Theorem 8 (the Maximum-Minimum Principle) to find the maximum over the domain. Read about the Maximum-Minimum principle on pages 236-237 and see example 1 and exercises 97-101 in Section 3.4 for more help.
>
> "What! How could I be wrong, I thought to myself, crestfallen. Do you see my mistake? Look carefully: I forgot to apply the last step of the process: to answer the question I was asked. I was ***not*** asked to find the value of *t* where the function achieves its maximum. Rather, I was told, "Find the maximum productivity." That is, I was supposed to answer with *M(t)* where *t = 30*. The correct answer was 1970, which is *M(30)* of course, not 30.
>
> "So, by making my mistake, I nonetheless made my point — answer the question you're asked!"

On Wednesday this week we'll hold the review for Test #2. Remember, the material in Section 2.6 has been deferred until later and will not be on the test (because of the time lost due to the recent flooding). Also, we will not cover Section 2.7 at all. The practice test is available, so be sure to start working on it and bring questions to the review tomorrow.

Added after Day 17 class meeting, Wednesday, October 14, 2009

The review for Monday's test was held today.

Added after Day 18 class meeting, Monday, October 19, 2009

The test was held today.

Added after Day 19 class meeting, Wednesday, October 21, 2009

Business is always interested in keeping track of marginal costs, revenue, and profit … that is, the cost (or revenue or profit) for just one more of the things being produced. This is usually very close to the value of the derivative function using the current production number. In fact, it is so close to being right, that many people just go ahead and call the **derivative** and **marginal** cost (or revenue, or profit) one and the same.

Recall that the derivative is defined as the limit of the difference quotient as the denominator approaches the value of zero: for small values of Δx (some authors will use the letter "h" instead of Δx):

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \text{ means } g'(x) \approx \frac{g(x+\Delta x) - g(x)}{\Delta x}.$$

Remember those important words: for small values of Δx. That wavy-looking equals-sign means "almost equal to."

From this, we solve and get the result that $g(x) + \Delta x * g'(x) \approx g(x+\Delta x)$. (Just multiply both sizes of that "almost equation" by Δx and then add $g(x)$ to both sides and you get that result.)

This says, in other words, that an output value of the function just a little bit (Δx) in the future will be almost equal to the previous output value, plus the amount gotten by multiplying the derivative at the previous value by the small additional amount being added to "x". Or, said another way, the tangent line hugs the function's graph so closely (at least for a little while) that we can safely use it to estimate the height of the function itself.

In a business situation, "profit" for instance, the additional amount is set to the value of 1, so that the derivative at the previous value is a reasonable estimate of the additional amount to be added to the function's previous output value to get the next output value; so, for instance, the "marginal profit" (the additional profit attributable to one more unit being produced) is nicely approximated by the value of the derivative of the profit function for the current production level. In short, the marginal profit (the profit for "one more" unit) at the current production level is equal to the derivative at that production level.

Put another way $\Delta y \approx \Delta x \cdot y'$. The exercises assigned for this section all are intended to show just that: the actual difference in the output values (the Δy) is almost exactly duplicated by multiplying the derivative by Δx.

Here's an example. Suppose you were desperate to have a good approximation for the value of the cube-root of 127: $\sqrt[3]{127}$. You know it must be just a bit more than the cube-root of 125, which is exactly equal to 5 (since 5^3=125). But, how much more than 5 is it?

Notice that in this case the function is the cube-root function: $f(x) = x^{1/3}$, and that *x*=125, while $\Delta x = 2$. So, that would mean $x + \Delta x = 125 + 2 = 127$. And hence, $\sqrt[3]{127} = f(x+\Delta x) \cong f(x) + \Delta x \cdot f'(x) = \sqrt[3]{125} + 2 \cdot f'(125)$. To complete that calculation, you need to have determined that in this case, since $f(x) = x^{1/3}$, then $f'(x) = \left(\tfrac{1}{3}\right)x^{-2/3} = \dfrac{1}{3\left(\sqrt[3]{x}\right)^2}$ and so $f'(125) = \dfrac{1}{3\left(\sqrt[3]{125}\right)^2} = \dfrac{1}{3 \cdot 5^2} = \dfrac{1}{75}$. Therefore, $\sqrt[3]{127} = f(x+\Delta x) \cong f(x) + \Delta x \cdot f'(x) = \sqrt[3]{125} + 2 \cdot \tfrac{1}{75} = 5 + \tfrac{2}{75} \cong 5.026666667$. How does this compare with the actual value of $\sqrt[3]{127}$? My TI-83 says it's equal to 5.026525695. Rounding both figures to 4 decimal places, they're pretty close: the approximation of 5.0267 is close to 5.0265, after all.

For our next class, we'll press forward into Chapter 3.