MATH 1101/24 — Mathematical Modeling — Fall 2009 — Instructor: Bruce Thomas
Practice Test #1

Instructions: Round answers to 2 decimal places, but if the context of the question clearly calls for an answer that could only be a whole number, then round up to the next whole number. In most of the following questions, if you simply answer with a number or an algebraic expression, that will be insufficient and points will be deducted. As has been told to you in class: <u>A number without any words to describe its meaning is, for the most part, meaningless and worthless!</u> Similarly, algebraic expressions floating by themselves are invalid; they should be part of an equation, with words appended that completely explain the meaning of variable names, with appropriate units of measurement appended. Use words with the numbers to communicate the full meaning of an answer. <u>You must show the calculations that support your answers</u>. The point-value for each question is shown in parentheses after the question number.

1. (6) It has been determined that the size of an animal population is described by the function $N(t) = 0.5 + 2t \times 2^{-t}$, where t is measured in years and N is measured in thousands of animals.

(a) Calculate $N(4.5)$ and explain its meaning.

N(4.5) = 0.90 — meaning that after 4.5 years, there were about 900 animals in the population.

(b) What is the initial population size?

N(0) = 0.5 — meaning that initially there were about 500 animals in the population.

(c) Use functional notation to express the population size after 18 months, and then calculate that population size.

N(1.5) = 1.56 — meaning that after 1.5 years (18 months), there were about 1,560 animals in the population.

2. (6) The following is a partial table of values for $f = f(x)$.

x	0	3	6
$f = f(x)$	50	55	61

(a) Find the average rate of change for f between $x = 0$ and $x = 3$.

The average rate of change is $\frac{55-50}{3-0} \cong 1.67$

(b) Find the average rate of change for f between $x = 3$ and $x = 6$.

The average rate of change is $\frac{61-55}{6-3} = \frac{6}{3} = 2$

(c) Use your answer to part (b) to estimate the value of $f(4)$.

Since the rate of change in that case is 2, this means that every increase of 1 in the x direction will produce a change of 2 in the vertical direction. Thus, $f(4) = 55 + 1 \times 2 = 57$.

3. (4) Suppose that *f* is a linear function such that *f*(3) = 5 and *f*(7) = -4. Use this information to write the the equation for *f* in the slope-intercept form of an equation. (Leave the slope and y-intercept info as fractions.)

The two points described have the coordinates of (3, 5) and (7, -4). Therefore, using the difference quotient again, the slope ("average rate of change") is simply $\frac{-4-5}{7-3} = -\frac{9}{4}$. Then using the point-slope form of the equation of a straight line, $y-5 = -\frac{9}{4}(x-3) = -\frac{9}{4}x + \frac{27}{4}$. Adding 5 (or 20/4) to both sides of that brings the equation into the slope-intercept form of $y = -\frac{9}{4}x + \frac{47}{4}$.

4. (18) A trainee is hired to test personal computers as they come off the assembly line. The learning curve for an average trainee is given by

$$N = \frac{200}{4+21e^{-0.1t}},$$

where N is the number of computers tested per day, and *t* is days on the job. (**Hint**: *If you have correctly entered this into your calculator, then a value of 5 for the t will result in a value of about 11.95 for N.*)

(a) Make a table showing the number of computers tested per day for 0,10, 20, and 30 days on the job.

t	*N*
0	8.00
10	17.06
20	29.23
30	39.64

(b) Use functional notation to express the number of computers tested per day after 25 days on the job and then calculate its value.

$N(25) \cong 34.94 \cong 35$ radios tested per day after 25 days on the job.

(c) What is the average rate of change in the learning curve from 20 to 25 days? From 25 to 30 days?

$\frac{34.94-29.23}{25-20} \cong 1.14$ additional radios (avg) per day from day 20 to day 25

$\frac{39.64-34.94}{30-25} \cong 0.94$ additional radios (avg) per day from day 25 to day 30

(d) Explain what your answers in Part (c) tell you about the growth in the number of computers tested per day.

While the trainee's productivity continues to grow, that growth appears to be slowing down.

(e) What is the limiting value for this function? Explain how you know.

Using a graph, as *t* increases the curve seems to be leveling off at about 50.

(f) Write a sentence that interprets the meaning of the limiting value in part (e) above.

The most number of radios a trainee will be able to test in a day is limited to 50.

5. (8) The weekly cost C (in dollars) of running a telemarketing firm is related to the number N of telephone operators. There is a fixed cost of $1500 per week, and each telephone operator costs the firm $300 per week. For example, if there are 10 operators, then the weekly cost is $4500.

(a) Use a formula to express the weekly cost C as a function of N, the number of telephone operators.

> $C = 1500 + 300N$ dollars of weekly cost, where N is the number of telephone operators.

(b) Explain in practical terms what C(8) means, and then use your formula from Part (a) to calculate it.

> $C(8)$ is the weekly cost when there are 8 telephone operators employed.
> $C(8) = 1500 + 300 * 8 =$ $3,900.

(c) Use functional notation to express the weekly cost if 5 telephone operators are used, and then use your formula from Part (a) to calculate that cost.

> $C(5)$ is the weekly cost when there are 5 telephone operators employed.
> $C(5) = 1500 + 300 * 5 =$ $3,000.

(d) The costs of running the firm have now increased, and the new weekly cost D (in dollars) is given by the formula D = 1550 + 350N, where N is the number of telephone operators. How much does each telephone operator cost the firm now?

> It should be clear that each telephone operator now costs the telemarketing firm $350 per week.

6. (4) You pay $500 to rent a special area in a restaurant. In addition, you pay $10 for each guest. Write a formula that gives your total cost *C*, in dollars, as a function of the number *n* of dinner guests.

> $C = C(n) = 500 + 10n$ dollars is the cost to rent the special area and entertain n guests.

7. (8) Banks and other lending institutions typically require that their loans be paid back in monthly installments all of the same size. The amount of the monthly loan payment, *M*, depends on three things: the amount that was borrowed, *P*; the monthly interest rate (expressed as a decimal), *r*; and the number of months over which the loan will be totally repaid, *m*. The formula to calculate the monthly loan payment is

$$M = \frac{P \times r \times (1+r)^m}{(1+r)^m - 1} \quad .$$

Use functional notation to express the monthly loan payment for each the following loans, and then use the given formula to calculate the amount of the monthly loan payment:

(a) A car was purchased for $22,495 and the entire amount was borrowed from a bank at a monthly interest rate of 0.75%, with monthly payments extending for a full 5 years.

M = *M*(*P*,*r*,*m*) dollars is the monthly payment on a loan for *P* dollars, at a monthly interest rate of 100**r* %, over a loan period of *m* months. In this instance, the monthly payment amount is *M*(22495, 0.0075, 60) = $466.96

(b) A young couple is about to purchase their first house. The agreed upon purchase price is $245,495 and they plan to make a down payment equal to 10% of the purchase price, and borrow the remainder of the purchase price. Their lender has agreed to lend them what they'll need to borrow at an annual percentage rate of 6%, with monthly payments over a 30 year period.

M = *M*(*P*,*r*,*m*) dollars is the monthly payment on a loan for *P* dollars, at a monthly interest rate of 100**r* %, over a loan period of *m* months. In this instance, the monthly payment amount is *M*(245495, 0.005, 360) = $1,471.90 [since the annual percentage rate is 6%, the monthly rate is 1/12th of that or 0.5% which, in decimal terms, is 0.005]

8. (8) The person who purchased the car in part (a) of question (7) above is curious to see how much his monthly payments would have been if he had decided on a different length of time for the loan. Create a table below that shows the monthly loan payment amount for various lengths of time, starting with 3 years and ending with 5 years, with increments of 6 months between each table entry.

For question #8

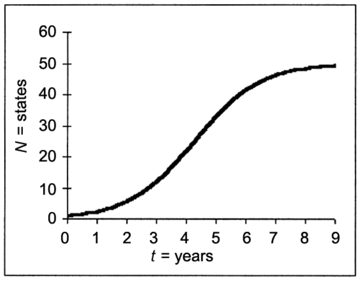
m	*M*
36	$715.33
42	$626.36
48	$559.76
54	$508.15
60	$466.96

For question #9

m	*M*	Total paid
36	$715.33	$25,752.06
42	$626.36	$26.307.22
48	$559.76	$26,869.87
54	$508.15	$27,439.99
60	$466.96	$28,017.55

9. (4) Enhance the table you created in question (8) so that it also shows the total amount that will be paid to the lender over the full length of the loan in each case.

10. (18) In recent years many states have passed mandatory seat belt laws. The total number *N* of states that had a seat belt law in effect *t* years after 1984 is given by the graph below.



(a) How many states had a seat belt law in effect in 1989?

The graph indicates that about 33 states had a seat beat in effect in 1989.

(b) In what year were there seat belt laws in effect in 40 states?

The graph indicates that by 1990 40 states had a seat beat in effect.

(c) When was the graph concave up? Explain how the number of states with seat belt laws was changing during this period.

From 1984 until 1989-1990 the graph was concave up. This shows that during that period the growth in the number of states with a seat belt law in effect was accelerating.

(d) When was the graph concave down? Explain how the number of states with seat belt laws was changing during this period.

From 1989-1990 on the graph was concave down. This shows that during from that point on the growth in the number of states with a seat belt law in effect was decelerating, even though the number was still growing.

(e) Between which two years was the time of fastest growth? What name is given in mathematics to the spot on a graph where the rate of change is the greatest?

Sometime during the years 1989-1990 the growth was the fastest. The point of fastest growth occurred where the inflection point occurs.

(f) Does the function represented by this graph have a limiting value? What is it? Explain the reasoning you applied to arrive at your answer.

It is pretty clear that this is a logistic curve and that it does, indeed, have a limiting value. The limiting value appears to be 50 — which is exactly how many states there are!

11. (16) A company buys a new color photocopy machine for \$6,180. The machine decreases in value (depreciates) by \$515 per year. The value V of the machine is a function of time t in years since the machine was purchased.

(a) Make a table of values to show the value of the machine for 0 to 4 years.

t	0	1	2	3	4
V	\$6,180	\$5,665	\$5,150	\$4,635	\$4,120

(b) Use a formula to express the value V as a function of t.

$V = V(t) = 6180 - 515t$ dollars of value,
where t is the number of years since the machine was purchased.

(c) How long will it take for the machine to lose half its original value?

Since half of \$6,180 is \$3,090, solve this equation for t to determine the year when the value will have been cut in half: $3090 = 6180 - 515t$. The answer is $t = 6$. Therefore it will take 6 years for the value of the machine to be reduced to half of its original purchase price.

(d) How long will it take for the machine to depreciate to the point where it has no residual value?

Since it took 6 years for the machine to lose half of its value, and since this is a steady ("straight line depreciation") annual decrease, it will obviously take 12 years to reach the point where there is no residual value. Or, you can just calculate it directly by solving this linear equation: $0 = 6180 - 515t$.